General notations

Functions

By default, the functions that we consider are complex-valued.

The space variable, denoted by x, belongs to $\mathbb{R}^n$. The time variable is denoted by t.

The partial derivatives with respect to the time variable and to the j-th space variable are denoted by ∂_t and ∂_j, respectively.

We denote by Λ the Fourier multiplier $(\mathrm{Id} - \Delta)^{1/2}$, where Δ stands for the Laplacian

$$\Delta = \sum_{j=1}^{n} \partial_j^2.$$

Function spaces

We denote by $L^p(\mathbb{R}^n)$, or simply L^p, the usual Lebesgue spaces on $\mathbb{R}^n$.
The inner product of $L^2(\mathbb{R}^n)$ is defined as

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)\overline{g}(x)dx.$$

Consider $f = f(t,x)$ a function from $I \times \mathbb{R}^n$ to $\mathbb{C}$, where I is a time interval.
If $f \in C(I; L^p(\mathbb{R}^n))$, we write

$$\|f\|_{L^\infty(I;L^p)} = \sup_{t \in I} \|f(t)\|_{L^p(\mathbb{R}^n)}.$$

The Schwartz class of smooth functions $\mathbb{R}^n \to \mathbb{C}$ which decay rapidly as well as all their derivatives is denoted by $\mathcal{S}(\mathbb{R}^n)$.

For $f \in \mathcal{S}(\mathbb{R}^n)$, we define its Fourier transform by

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx,$$

so that the inverse Fourier transform is given by

$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

For $s \geqslant 0$, we define the Sobolev space $H^s(\mathbb{R}^n) = H^s$ as

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \ ; \ \xi \mapsto \langle \xi \rangle^s \, \widehat{f}(\xi) \in L^2(\mathbb{R}^n) \right\},$$

where we have denoted $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$. Note that if $s \in \mathbb{N}$, then

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \ ; \ \partial^\alpha f \in L^2(\mathbb{R}^n), \ \forall \alpha \in \mathbb{N}^n, \ |\alpha| \leqslant s \right\}.$$

Recall that if $s > n/2$, then $H^s(\mathbb{R}^n)$ is an algebra, and $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$.

The set $H^\infty(\mathbb{R}^n)$, or simply H^∞, is the intersection of all these spaces:

$$H^\infty = \cap_{s \geqslant 0} H^s(\mathbb{R}^n).$$

This is a Fréchet space, equipped with the distance

$$d(f,g) = \sum_{s \in \mathbb{N}} 2^{-s} \frac{\|f-g\|_{H^s}}{1+\|f-g\|_{H^s}}.$$

Semi-classical limit

The dependence of functions upon the semi-classical parameter ε is denoted by a superscript. For instance, the wave function is denoted by u^ε.

All the irrelevant constants are denoted by C. In particular, C stands for a constant which is independent of ε, the semi-classical parameter.

Let $(\alpha^h)_{0<h\leqslant 1}$ and $(\beta^h)_{0<h\leqslant 1}$ be two families of positive real numbers.

- We write $\alpha^h \ll \beta^h$, or $\alpha^h = o(\beta^h)$, if $\limsup_{h\to 0} \alpha^h/\beta^h = 0$.
- We write $\alpha^h \lesssim \beta^h$, or $\alpha^h = \mathcal{O}(\beta^h)$, if $\limsup_{h\to 0} \alpha^h/\beta^h < \infty$.
- We write $\alpha^h \approx \beta^h$ if $\alpha^h \lesssim \beta^h$ and $\beta^h \lesssim \alpha^h$.

If u^h and v^h are functions, we write $u^h \approx v^h$ if $\|u^h - v^h\| \ll \|v^h\|$, for some norm to be precised (or not, when computations are purely formal).

Contents

PART 1
WKB Analysis

Chapter 1

Preliminary analysis

We consider nonlinear Schrödinger equations in the presence of a parameter $\varepsilon \in]0,1]$,

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + f\left(|u^\varepsilon|^2\right)u^\varepsilon, \tag{1.1}$$

where $u^\varepsilon = u^\varepsilon(t,x)$ is complex-valued. Throughout this book, the space variable, denoted by x, lies in the whole Euclidean space $\mathbb{R}^n$, $n \geqslant 1$. Many of the results presented in this first part can easily be adapted to the case of the torus $\mathbb{T}^n$. The external potential $V = V(t,x)$ and the (local) nonlinearity f are supposed to be smooth, *real-valued*, and independent of ε. The aim of these notes is to describe some results about the asymptotic behavior of the solution u^ε as the parameter ε goes to zero. We shall be more precise about the initial data that we consider below. The nonlinearity f is *local* (e.g. power-like nonlinearity): in particular, we choose not to mention results related to nonlocal nonlinearities, such as the Schrödinger–Poisson system

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + V_p u^\varepsilon \quad ; \quad \Delta V_p = \lambda\left(|u^\varepsilon|^2 - c\right),$$

or the Hartree equation

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + \lambda\left(\frac{1}{|x|^\gamma} * |u^\varepsilon|^2\right)u^\varepsilon.$$

We do not consider ε-dependent potential either, an issue for which the main model we have in mind is that of a lattice periodic potential, whose period is of order ε:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + V_\Gamma\left(\frac{x}{\varepsilon}\right)u^\varepsilon + f\left(|u^\varepsilon|^2\right)u^\varepsilon,$$

where the potential V_Γ is periodic with respect to some regular lattice $\Gamma \simeq \mathbb{Z}^n$. See for instance [Bensoussan *et al.* (1978); Robert (1998); Teufel

(2003)] for an introduction to the asymptotic study in the linear case of the above equation, and [Carles *et al.* (2004)] for an example of asymptotic behavior in a nonlinear régime. Our choice is to focus on (1.1), and to describe as precisely as possible the variety of known phenomena in the limit $\varepsilon \to 0$.

There are several reasons to study the asymptotic behavior of u^ε in the semi-classical limit $\varepsilon \to 0$. Let us mention two. First, (1.1) with $f(|u|^2)u = |u|^4 u$ (quintic nonlinearity) is sometimes used as a model for one-dimensional Bose–Einstein condensation in space dimension $n = 1$ ([Kolomeisky *et al.* (2000)]). When $n = 2$ or 3, a cubic nonlinearity, $f(|u|^2)u = |u|^2 u$, is usually considered. The external potential V can be an harmonic potential (isotropic or anisotropic), or a lattice periodic potential (see e.g. [Dalfovo *et al.* (1999); Pitaevskii and Stringari (2003)]). According to the different physical parameters at stake, the asymptotic behavior of u^ε as $\varepsilon \to 0$ may provide relevant informations to describe u^ε itself. This approach is similar to the theory of geometric optics, developed initially to describe the propagation of electro-magnetic waves, such as light. In that context, the propagation of the wave is also described by partial differential equations, and ε usually corresponds to a wavelength, which is small compared to the other parameters. For Maxwell's equations, ε corresponds to the inverse of the speed of light. We invite the reader to consult [Rauch and Keel (1999)] for an overview of this theory, mainly in the context of hyperbolic equations. We shall not develop further on the physical motivations, but rather focus our attention on the mathematical aspects. The term "geometric optics" means that it is expected that the propagation of light is accurately described by rays. For Schrödinger equations, the analogue of this notion is usually called "classical trajectories". These notions are identical, and follow from the notion of bicharacteristic curves. As a consequence, the limit $\varepsilon \to 0$ relates classical and quantum wave equations. In particular, the semi-classical limit $\varepsilon \to 0$ for u^ε is expected to be described by the laws of hydrodynamics. We will come back to this aspect more precisely later.

Another motivation lies in the study the Cauchy problem for nonlinear Schrödinger equations with no small parameter ($V \equiv 0$ and $\varepsilon = 1$ in (1.1), typically). One can prove ill-posedness results for energy-supercritical equations by reducing the problem to semi-classical analysis for (1.1). This aspect is discussed in details in Sec. 5.1 and Sec. 5.2. Note that the application of the theory of geometric optics to functional analysis has a long history.

In [Lax (1957)], it was used to construct parametrices. It has also been used to study the propagation of singularities (see e.g. [Taylor (1981)]), or of quasi-singularities [Cheverry (2005)]. In the case of Schrödinger equations, semi-classical analysis has proven useful for instance in control theory [Lebeau (1992)], in the proof of Strichartz estimates [Burq *et al.* (2004)], and in the propagation of singularities for the nonlinear equation [Szeftel (2005)].

We underscore the fact that the WKB analysis for (nonlinear) Schrödinger equations is rather specific to this equation. An important feature is the fact that for gauge invariant nonlinearities, it is possible to describe the solution with one phase and one harmonic only, provided that the initial data are of this form: $u^\varepsilon \approx a e^{i\phi/\varepsilon}$. For several other equations (e.g. Maxwell equations), the analysis is rather different, even on the algebraic level. We invite the reader to consult for instance [Joly *et al.* (1996b); Métivier (2004b); Rauch and Keel (1999); Whitham (1999)], and references therein, to have an idea of the important results for equations different from the Schrödinger equation. However, the general framework presented in §1.1 (derivation of the equation, and steps toward a justification) is not specific to the equation: the main specificity of gauge invariant nonlinear Schrödinger equations (as in Eq. (1.1)) is that the equations derived at the formal step look simpler than for other equations, due to the fact that we work with only one phase (and one harmonic).

Before introducing the approach developed in this first part, we present two basic results, which will be used throughout these notes.

Lemma 1.1 (Gronwall lemma and a continuity argument).
(1) *Let* $u, a, b \in C([0,T];\mathbb{R}_+)$ *be such that*

$$u(t) \leqslant u(0) + \int_0^t a(\tau)u(\tau)d\tau + \int_0^t b(\tau)d\tau, \quad \forall t \in [0,T].$$

Denote $A(t) = \int_0^t a(\tau)d\tau$. *Then*

$$u(t) \leqslant u(0)e^{A(t)} + \int_0^t b(s)e^{A(t)-A(s)}ds, \quad \forall t \in [0,T].$$

(2) *Let* $u, b \in C([0,T];\mathbb{R}_+)$ *and* $f \in C(\mathbb{R}_+;\mathbb{R}_+)$ *such that*

$$u(t) \leqslant u(0) + \int_0^t f\left(u(\tau)\right)u(\tau)d\tau + \int_0^t b(\tau)d\tau, \quad \forall t \in [0,T].$$

Let $M = \sup\{f(v);\ v \in [0,2u(0)]\}$. *There exists* $\underline{t} \in]0,T]$ *such that*

$$u(t) \leqslant u(0)e^{Mt} + \int_0^t b(s)e^{M(t-s)}ds, \quad \forall t \in [0,\underline{t}].$$

Proof. (1) Denote

$$w(t) = u(0) + \int_0^t a(\tau)u(\tau)d\tau + \int_0^t b(\tau)d\tau.$$

By assumption, $w \in C^1([0,T])$ and $w'(t) = a(t)u(t)+b(t) \leqslant a(t)w(t)+b(t)$. Therefore,

$$\left(w(t)e^{-A(t)}\right)' \leqslant b(t)e^{-A(t)},$$

and the first point follows by integrating this inequality, since $u(t) \leqslant w(t)$.
(2) Suppose that there exists $t \in]0,T]$ such that $u(t) > 2u(0)$. Since u is continuous, we can define

$$\underline{t} = \min\{\tau \in [0,T];\ u(\tau) = 2u(0)\} > 0.$$

The assumption implies

$$u(t) \leqslant u(0) + M\int_0^t u(\tau)d\tau + \int_0^t b(\tau)d\tau, \quad \forall t \in [0,\underline{t}].$$

Gronwall lemma then yields

$$u(t) \leqslant u(0)e^{Mt} + \int_0^t b(s)e^{M(t-s)}ds, \quad \forall t \in [0,\underline{t}].$$

The right hand side is continuous, and is equal to $u(0)$ for $t = 0$. Up to decreasing $\underline{t}$, this right hand side does not exceed $2u(0)$ for $t \in [0,\underline{t}]$, hence the conclusion of the lemma.

If $u(t) \leqslant 2u(0)$ for all $t \in [0,T]$, then we can trivially take $\underline{t} = T$. □

Lemma 1.2 (Basic energy estimate). *For $\varepsilon > 0$, consider $\mathbf{u}^\varepsilon$ solving*

$$i\varepsilon\partial_t \mathbf{u}^\varepsilon + \frac{\varepsilon^2}{2}\Delta\mathbf{u}^\varepsilon = F^\varepsilon\mathbf{u}^\varepsilon + R^\varepsilon \quad ; \quad \mathbf{u}^\varepsilon_{|t=0} = \mathbf{u}^\varepsilon_0. \tag{1.2}$$

Assume that $F^\varepsilon = F^\varepsilon(t,x)$ is real-valued. Let I be a time interval such that $0 \in I$. Then we have, at least formally:

$$\sup_{t\in I}\|\mathbf{u}^\varepsilon(t)\|_{L^2} \leqslant \|\mathbf{u}^\varepsilon_0\|_{L^2} + \frac{1}{\varepsilon}\int_I \|R^\varepsilon(\tau)\|_{L^2}\,d\tau.$$

Proof. Since the statement is formal, so is the proof. Multiply (1.2) by $\overline{\mathbf{u}}^\varepsilon$, and integrate over $\mathbb{R}^n$:

$$i\varepsilon\int_{\mathbb{R}^n}\overline{\mathbf{u}}^\varepsilon\partial_t\mathbf{u}^\varepsilon dx + \frac{\varepsilon^2}{2}\int_{\mathbb{R}^n}\overline{\mathbf{u}}^\varepsilon\Delta\mathbf{u}^\varepsilon dx = \int_{\mathbb{R}^n}F^\varepsilon|\mathbf{u}^\varepsilon|^2dx + \int_{\mathbb{R}^n}\overline{\mathbf{u}}^\varepsilon R^\varepsilon dx.$$

Taking the imaginary part, the second term of the left hand side vanishes, since Δ is self-adjoint. Similarly, since F^ε is real-valued, the first term of the right hand side disappears, and we have:

$$\varepsilon \frac{d}{dt}\int_{\mathbb{R}^n} |\mathbf{u}^\varepsilon|^2 = \varepsilon \int_{\mathbb{R}^n} \partial_t |\mathbf{u}^\varepsilon|^2 = 2\,\mathrm{Im}\int_{\mathbb{R}^n} \overline{\mathbf{u}}^\varepsilon R^\varepsilon .$$

Cauchy–Schwarz inequality yields

$$\varepsilon \frac{d}{dt}\|\mathbf{u}^\varepsilon\|_{L^2}^2 \leqslant 2\|\mathbf{u}^\varepsilon\|_{L^2}\|R^\varepsilon\|_{L^2}.$$

Let $\delta > 0$. We infer from the above inequality:

$$\varepsilon \frac{d}{dt}\left(\|\mathbf{u}^\varepsilon\|_{L^2}^2 + \delta\right) \leqslant 2\left(\|\mathbf{u}^\varepsilon\|_{L^2}^2 + \delta\right)^{1/2}\|R^\varepsilon\|_{L^2}.$$

Since $\|\mathbf{u}^\varepsilon\|_{L^2}^2 + \delta \geqslant \delta > 0$, we can simplify:

$$\varepsilon \frac{d}{dt}\left(\|\mathbf{u}^\varepsilon\|_{L^2}^2 + \delta\right)^{1/2} \leqslant \|R^\varepsilon\|_{L^2}.$$

Integration with respect to time yields, for $t \in I$:

$$\varepsilon\left(\|\mathbf{u}^\varepsilon(t)\|_{L^2}^2 + \delta\right)^{1/2} \leqslant \varepsilon\left(\|\mathbf{u}_0^\varepsilon\|_{L^2}^2 + \delta\right)^{1/2} + \int_I \|R^\varepsilon(t)\|_{L^2} dt.$$

The lemma follows by letting $\delta \to 0$. □

1.1 General presentation

The general approach of WKB expansions (after three papers by Wentzel, Kramers and Brillouin respectively, in 1926) consists of mainly three steps. The first step, which is described in more details in this section, consists in seeking a function v^ε that solves (1.1) up to a small error term:

$$i\varepsilon\partial_t v^\varepsilon + \frac{\varepsilon^2}{2}\Delta v^\varepsilon = V v^\varepsilon + f\left(|v^\varepsilon|^2\right) v^\varepsilon + r^\varepsilon,$$

where r^ε should be thought of as a "small" (as $\varepsilon \to 0$) source term. Typically, we require

$$\|r^\varepsilon\|_{L^\infty([-T,T];L^2)} = \mathcal{O}\left(\varepsilon^N\right)$$

for some $T > 0$ independent of ε, and $N > 0$ as large as possible. In this first step, we derive equations that define v^ε, which are hopefully simpler than (1.1). The second step consists in showing that such a v^ε actually exists, that is, in solving the equations derived in the first step. The last step is the study of *stability* (or *consistency*): even if r^ε is small, it is not clear that $u^\varepsilon - v^\varepsilon$ is small too. Typically, we try to prove an error estimate of the form

$$\|u^\varepsilon - v^\varepsilon\|_{L^\infty([-T,T];L^2)} = \mathcal{O}\left(\varepsilon^K\right)$$

for some $K > 0$ (possibly smaller than N). Note also that for the nonlinear problem (1.1), it is not even clear from the beginning that an L^2 solution can be constructed on a time interval independent of $\varepsilon \in]0,1]$.

The initial data that we consider for WKB analysis are of the form

$$u^\varepsilon(0,x) = \varepsilon^\kappa a_0^\varepsilon(x) e^{i\phi_0(x)/\varepsilon}. \tag{1.3}$$

The phase ϕ_0 is independent of ε and real-valued. The initial amplitude a_0^ε is complex-valued, and may have an asymptotic expansion as $\varepsilon \to 0$,

$$a_0^\varepsilon(x) \underset{\varepsilon\to 0}{\sim} a_0(x) + \varepsilon a_1(x) + \varepsilon^2 a_2(x) + \dots, \tag{1.4}$$

in the sense of formal asymptotic expansions, where the profiles a_j are independent of ε. Note that ε^κ then measures the size of $u^\varepsilon(0,x)$ in $L^\infty(\mathbb{R}^n)$. We shall always consider cases where $\kappa \geqslant 0$. When the nonlinearity is non-trivial, $f \neq 0$, the asymptotic behavior of u^ε as $\varepsilon \to 0$ strongly depends on the value of κ, as is discussed below. An important feature of Schrödinger equations with gauge invariant nonlinearities like in (1.1) is that if the initial data are of the form (1.3), then for small time at least (before caustics), the solution u^ε is expected to keep the same form, at least approximately:

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \varepsilon^\kappa a^\varepsilon(t,x) e^{i\phi(t,x)/\varepsilon}, \tag{1.5}$$

where a^ε is expected to have an asymptotic expansion as well. This is in sharp contrast with the analogous problems for hyperbolic equations (e.g. Maxwell, wave, Euler): typically, because the solutions of the wave equations are real-valued, the factor $e^{i\phi_0/\varepsilon}$ is replaced, say, by $2\cos(\phi_0/\varepsilon) = e^{i\phi_0/\varepsilon} + e^{-i\phi_0/\varepsilon}$. By nonlinear interaction, other phases are expected to appear, like $e^{ik\phi/\varepsilon}$, $k \in \mathbb{Z}$, for instance. This can be guessed by looking at the first iterates of a Picard's scheme. Moreover, phases different from ϕ might be involved in the description of u^ε, by nonlinear mechanisms too. We will see that unlike for these models, such a phenomenon is ruled out for nonlinear Schrödinger equations, provided that only one phase is considered initially, see (1.3). This is an important geometric feature in this study. On the other hand, studying the asymptotic behavior of u^ε whose initial data are *sums* of initial data as in (1.3) is an interesting open question so far.

To describe the expected influence of the parameter κ on the asymptotic behavior of u^ε, assume that the nonlinearity f is homogeneous:

$$f\left(|u^\varepsilon|^2\right)u^\varepsilon = \lambda |u^\varepsilon|^{2\sigma}u^\varepsilon, \quad \lambda\in\mathbb{R},\ \sigma>0.$$

The case $\sigma\in\mathbb{N}\setminus\{0\}$ corresponds to a smooth nonlinearity. Even though the parameter κ may be viewed as a measurement of the size of the (initial) wave function, we shall rather consider data of order $\mathcal{O}(1)$, by introducing $\widetilde{u}^\varepsilon = \varepsilon^{-\kappa}u^\varepsilon$. Dropping the tildes, we therefore consider

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + \lambda\varepsilon^\alpha |u^\varepsilon|^{2\sigma}u^\varepsilon \quad ; \quad u^\varepsilon(0,x) = a_0^\varepsilon(x)e^{i\phi_0(x)/\varepsilon}, \tag{1.6}$$

where $\alpha = 2\sigma\kappa \geqslant 0$.

1.2 Formal derivation of the equations

Assuming that the initial data have an asymptotic expansion of the form (1.4), we seek $u^\varepsilon(t,x)\sim a^\varepsilon(t,x)e^{i\phi(t,x)/\varepsilon}$, with

$$a^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} a(t,x) + \varepsilon a^{(1)}(t,x) + \varepsilon^2 a^{(2)}(t,x) + \dots$$

We use the convention $a^{(0)} = a$. On a formal level at least, the general idea consists in plugging this asymptotic expansion into (1.6), and then ordering in powers of ε. The lowest powers are the ones we really want to cancel, and if we are left with some extra terms, we want to be able to consider them as small source terms in the limit $\varepsilon\to 0$ (by a perturbative analysis for instance). To summarize, we first find $b^{(0)}, b^{(1)}, \dots$, such that

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon - Vu^\varepsilon - \lambda\varepsilon^\alpha|u^\varepsilon|^{2\sigma}u^\varepsilon \underset{\varepsilon\to 0}{\sim} \left(b^{(0)} + \varepsilon b^{(1)} + \varepsilon^2 b^{(2)} + \dots\right)e^{i\phi/\varepsilon}.$$

Then we consider the equations $b^{(0)} = 0$, $b^{(1)} = 0$, etc. Note that this makes sense provided that $\alpha\in\mathbb{N}$, for otherwise, non-integer powers of ε appear in the above right hand side.

Denoting by ∂ a differentiation with respect to the time variable, or any space variable, we compute formally:

$$\begin{aligned}\partial u^\varepsilon \underset{\varepsilon\to 0}{\sim} \Big(& i\varepsilon^{-1}\left(a + \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \dots\right)\partial\phi \\ & + \partial a + \varepsilon\partial a^{(1)} + \varepsilon^2\partial a^{(2)} + \dots\Big)e^{i\phi/\varepsilon}.\end{aligned}$$

Similarly, for $1 \leqslant j \leqslant n$,

$$\begin{aligned}\partial_j^2 u^\varepsilon \underset{\varepsilon\to 0}{\sim} \Big(& -\varepsilon^{-2}\left(a+\varepsilon a^{(1)}+\varepsilon^2 a^{(2)}+\dots\right)(\partial_j\phi)^2 \\ & + i\varepsilon^{-1}\left(a+\varepsilon a^{(1)}+\varepsilon^2 a^{(2)}+\dots\right)\partial_j^2\phi \\ & + 2i\varepsilon^{-1}\left(\partial_j a+\varepsilon\partial_j a^{(1)}+\varepsilon^2\partial_j a^{(2)}+\dots\right)\partial_j\phi \\ & + \partial_j^2 a+\varepsilon\partial_j^2 a^{(1)}+\varepsilon^2\partial_j^2 a^{(2)}+\dots\Big)e^{i\phi/\varepsilon}.\end{aligned}$$

Ordering in powers of ε, we infer:

$$\begin{aligned}i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon \underset{\varepsilon\to 0}{\sim} \Bigg(& -\left(\partial_t\phi+\frac{1}{2}|\nabla\phi|^2\right)\left(a+\varepsilon a^{(1)}+\varepsilon^2 a^{(2)}+\dots\right) \\ & + i\varepsilon\left(\partial_t a+\nabla\phi\cdot\nabla a+\frac{1}{2}a\Delta\phi\right) \\ & + i\varepsilon^2\left(\partial_t a^{(1)}+\nabla\phi\cdot\nabla a^{(1)}+\frac{1}{2}a^{(1)}\Delta\phi-\frac{i}{2}\Delta a\right) \\ & \ \vdots \\ & + i\varepsilon^{j+1}\left(\partial_t a^{(j)}+\nabla\phi\cdot\nabla a^{(j)}+\frac{1}{2}a^{(j)}\Delta\phi-\frac{i}{2}\Delta a^{(j-1)}\right) \\ & + \dots\Bigg)e^{i\phi/\varepsilon}.\end{aligned}$$

For the nonlinear term, we choose to compute only the first two terms:

$$|u^\varepsilon|^{2\sigma}u^\varepsilon \underset{\varepsilon\to 0}{\sim} \left(|a|^{2\sigma}a+\varepsilon\left(|a|^{2\sigma}a^{(1)}+2\sigma\operatorname{Re}\left(\overline{a}a^{(1)}\right)|a|^{2\sigma-2}a\right)+\dots\right)e^{i\phi/\varepsilon}.$$

To simplify the discussion, assume in the following lines that α is an integer, $\alpha\in\mathbb{N}$. Since we want to consider a leading order amplitude a which is not identically zero, it is natural to demand, for the term of order ε^0:

$$\partial_t\phi+\frac{1}{2}|\nabla\phi|^2+V=\begin{cases}0 & \text{if } \alpha>0,\\ -\lambda|a|^{2\sigma} & \text{if } \alpha=0.\end{cases} \tag{1.7}$$

For the term of order ε^1, we find:

$$\partial_t a+\nabla\phi\cdot\nabla a+\frac{1}{2}a\Delta\phi=\begin{cases}0 & \text{if } \alpha>1,\\ -i\lambda|a|^{2\sigma}a & \text{if } \alpha=1,\\ -2i\lambda\sigma\operatorname{Re}\left(\overline{a}a^{(1)}\right)|a|^{2\sigma-2}a & \text{if } \alpha=0.\end{cases} \tag{1.8}$$

Before giving a rigorous meaning to this approach, we comment on these cases. Intuitively, the larger the α, the smaller the influence of the nonlinearity: for large α, the nonlinearity is not expected to be relevant at leading order as $\varepsilon \to 0$. In terms of the problem (1.1)–(1.3), this means that small initial waves (large κ) evolve linearly at leading order: this corresponds to the general phenomenon that very small nonlinear waves behave linearly at leading order. Here, we see that if $\alpha > 1$, then ϕ and a solve equations which are independent of λ, hence of the nonlinearity. Since at leading order, we expect

$$u^\varepsilon(t,x) \underset{\varepsilon \to 0}{\sim} a(t,x) e^{i\phi(t,x)/\varepsilon},$$

this means that the leading order behavior of u^ε is linear. As a consequence, we also expect

$$u^\varepsilon(t,x) \underset{\varepsilon \to 0}{\sim} u^\varepsilon_{\rm lin}(t,x),$$

where $u^\varepsilon_{\rm lin}$ solves the linear problem

$$i\varepsilon\partial_t u^\varepsilon_{\rm lin} + \frac{\varepsilon^2}{2}\Delta u^\varepsilon_{\rm lin} = V u^\varepsilon_{\rm lin} \quad ; \quad u^\varepsilon_{\rm lin}(0,x) = u^\varepsilon(0,x) = a_0^\varepsilon(x) e^{i\phi_0(x)/\varepsilon}.$$

Decreasing the value of α, the critical threshold corresponds to $\alpha = 1$: the nonlinearity shows up in the equation for a, but not in the equation for ϕ. This régime is referred to as *weakly nonlinear geometric optics*. The term "weakly" means that the phase ϕ is determined independently of the nonlinearity: the equations for a and ϕ are decoupled. We will see that for $\alpha \geqslant 1$, the equation for a can be understood as a transport equation along the classical trajectories (rays of geometric optics) associated to ϕ, which in turn are determined by the initial phase ϕ_0 and the semi-classical Hamiltonian

$$\tau + \frac{|\xi|^2}{2} + V(t,x).$$

See Sec. 1.3.1 below.

The case $\alpha = 0$ is supercritical, and contains several difficulties. We point out two of those, which show that dealing with the supercritical case requires a different approach. First, the equation for the phase involves the amplitude a. But to solve the equation for a, it seems necessary to know $a^{(1)}$. One could continue the expansion in powers of ε at arbitrarily high order: no matter how many terms are included, the system is never closed. This aspect is a general feature of supercritical geometric optics (see also [Cheverry (2005, 2006); Cheverry and Guès (2007)]). The second difficulty concerns the *stability* analysis. We have claimed that the general approach consists in computing $\phi, a, a^{(1)}, \ldots$, so that

$$u_\ell^\varepsilon(t,x) := \Big(a(t,x) + \varepsilon a^{(1)}(t,x) + \ldots + \varepsilon^\ell a^{(\ell)}(t,x)\Big) e^{i\phi(t,x)/\varepsilon}$$

solves (1.6) up to a small error term. Typically (recall that $\alpha = 0$),

$$i\varepsilon\partial_t u_\ell^\varepsilon + \frac{\varepsilon^2}{2}\Delta u_\ell^\varepsilon = V u_\ell^\varepsilon + \lambda |u_\ell^\varepsilon|^{2\sigma} u_\ell^\varepsilon + \varepsilon^\ell r_\ell^\varepsilon,$$

where r_ℓ^ε is bounded in a space "naturally" associated to the study of (1.6). When working in spaces based on the conservation of the L^2 norm for nonlinear Schrödinger equations (see Sec. 1.4), we expect estimates in $L^\infty([0,T];L^2(\mathbb{R}^n))$ for some $T > 0$ independent of $\varepsilon \in]0,1]$. Suppose that we have managed to construct such an approximate solution u_ℓ^ε. Assume for simplicity that u^ε and u_ℓ^ε coincide at time $t = 0$. Setting $w_\ell^\varepsilon = u^\varepsilon - u_\ell^\varepsilon$, we have:

$$i\varepsilon\partial_t w_\ell^\varepsilon + \frac{\varepsilon^2}{2}\Delta w_\ell^\varepsilon = V w_\ell^\varepsilon + \lambda\left(|u^\varepsilon|^{2\sigma}u^\varepsilon - |u_\ell^\varepsilon|^{2\sigma}u_\ell^\varepsilon\right) - \varepsilon^\ell r_\ell^\varepsilon.$$

Suppose that u^ε and u_ℓ^ε remain bounded in $L^\infty(\mathbb{R}^n)$ on the time interval $[0,T]$. Then we have:

$$\big||u^\varepsilon(t,x)|^{2\sigma}u^\varepsilon(t,x) - |u_\ell^\varepsilon(t,x)|^{2\sigma}u_\ell^\varepsilon(t,x)\big| \leqslant C(T)|w_\ell^\varepsilon(t,x)|,$$

uniformly for $t \in [0,T]$ and $x \in \mathbb{R}^n$. Lemma 1.2 yields the formal estimate, for $t \in [0,T]$:

$$\varepsilon\|w_\ell^\varepsilon(t)\|_{L^2} \leqslant 2|\lambda|C(T)\int_0^t \|w_\ell^\varepsilon(\tau)\|_{L^2}d\tau + 2\varepsilon^\ell \int_0^t \|r_\ell^\varepsilon(\tau)\|_{L^2}d\tau.$$

Using Gronwall lemma, we infer:

$$\|w^\varepsilon(t)\|_{L^2} \leqslant C\varepsilon^{\ell-1}e^{Ct/\varepsilon}.$$

The exponential factor shows that this method may yield interesting results only up to time of the order $c\varepsilon|\log\varepsilon|^\theta$ for some $c,\theta > 0$. Note that in some functional analysis contexts, this may be satisfactory (see Sec. 5.1). However, in general, we wish to have a description of the solution of (1.6) on a time interval independent of ε.

In Chap. 9, we give a rather explicit example of a situation similar to the one considered above, where ℓ can be taken arbitrarily large, but w_ℓ^ε is not small in L^2, past the time where Gronwall lemma is satisfactory (see Sec. 9.1.3).

1.3 Linear Schrödinger equation

Before proceeding to the nonlinear analysis, we justify the above discussion in the linear case: we consider (1.6) with $\lambda = 0$, that is

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon \quad ; \quad u^\varepsilon(0,x) = a_0^\varepsilon(x)e^{i\phi_0(x)/\varepsilon}. \tag{1.9}$$

The results presented here will also be useful in the study of the pointwise behavior of the solution u^ε to (1.1) in the nonlinear case (see Chap. 2 and Sec. 4.2.2). We invite the reader to consult [Robert (1987)] for results related to the semi-classical limit of Eq. (1.9) with a different point of view.

1.3.1 *The eikonal equation*

To cancel the ε^0 term, the first step consists in solving (1.7):

$$\partial_t\phi_{\rm eik} + \frac{1}{2}|\nabla\phi_{\rm eik}|^2 + V = 0 \quad ; \quad \phi_{\rm eik}(0,x) = \phi_0(x). \tag{1.10}$$

This equation is called the *eikonal equation*. The term "eikonal" stems from the theory of geometric optics: the solution to this equation determines the set where light is propagated. In the case of the (linear) Schrödinger equation, we will see that a similar phenomenon occurs: the phase $\phi_{\rm eik}$ determines the way the initial amplitude a_0 is transported (see Sec. 1.3.2). Equation (1.10) is also referred to as a *Hamilton–Jacobi equation*. It is usually solved locally in space and time in terms of the semi-classical Hamiltonian

$$H(t,x,\tau,\xi) = \tau + \frac{|\xi|^2}{2} + V(t,x), \quad (t,x,\tau,\xi) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n.$$

More general Hamilton–Jacobi equations are equations of the form

$$H\left(t,x,\partial_t\phi,\nabla\phi\right) = 0,$$

where H is a smooth real-valued function of its arguments. For the propagation of light in a medium of variable speed of propagation $c(x)$, we have

$$H(t,x,\tau,\xi) = \tau^2 - c(x)^2|\xi|^2.$$

The local resolution of such equations appears in many books (see e.g. [Dereziński and Gérard (1997); Grigis and Sjöstrand (1994); Evans (1998)]), so we shall only outline the usual approach. Since in our case $\partial_\tau H = 1$, the Hamiltonian flow is given by the system of ordinary differential equations

$$\begin{cases} \partial_t x(t,y) = \partial_\xi H = \xi\left(t,y\right) & ; \quad x(0,y) = y, \\ \partial_t\xi(t,y) = -\partial_x H = -\nabla_x V\left(t,x(t,y)\right) & ; \quad \xi(0,y) = \nabla\phi_0(y). \end{cases} \tag{1.11}$$

The projection of the solution (x,ξ) on the physical space, that is $x(t,y)$, is called *classical trajectory*, or *ray*. The Cauchy-Lipschitz Theorem yields:

Lemma 1.3. *Assume that V and ϕ_0 are smooth: $V \in C^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})$ and $\phi_0 \in C^\infty(\mathbb{R}^n;\mathbb{R})$. Then for all $y \in \mathbb{R}^n$, there exists $T_y > 0$ and a unique solution to* (1.11)*, $(x(t,y),\xi(t,y)) \in C^\infty([-T_y,T_y]\times\mathbb{R}^n;\mathbb{R}^n)^2$.*

The link with (1.10) appears in

Lemma 1.4. *Let $\phi_{\rm eik}$ be a smooth solution to* (1.10)*. Then necessarily,*

$$\nabla\phi_{\rm eik}\left(t,x(t,y)\right) = \xi(t,y),$$

as long as all the terms remain smooth.

Proof. For $\phi_{\rm eik}$ a smooth solution to (1.10), introduce the ordinary differential equation

$$\frac{d}{dt}\widetilde{x} = \nabla\phi_{\rm eik}\left(t,\widetilde{x}\right) \quad ; \quad \widetilde{x}\big|_{t=0} = y. \tag{1.12}$$

By the Cauchy-Lipschitz Theorem, (1.12) has a smooth solution $\widetilde{x} \in C^\infty([-\widetilde{T}_y,\widetilde{T}_y])$ for some $\widetilde{T}_y > 0$ possibly very small. Set

$$\widetilde{\xi}(t) := \nabla\phi_{\rm eik}\left(t,\widetilde{x}(t)\right).$$

We compute

$$\begin{aligned}
\frac{d}{dt}\widetilde{\xi} &= \nabla\partial_t\phi_{\rm eik}\left(t,\widetilde{x}(t)\right) + \nabla^2\phi_{\rm eik}\left(t,\widetilde{x}(t)\right)\cdot\frac{d}{dt}\widetilde{x}(t)\\
&= \nabla\partial_t\phi_{\rm eik}\left(t,\widetilde{x}(t)\right) + \nabla^2\phi_{\rm eik}\left(t,\widetilde{x}(t)\right)\cdot\nabla\phi_{\rm eik}\left(t,\widetilde{x}(t)\right)\\
&= \nabla\left(\partial_t\phi_{\rm eik} + \frac{1}{2}|\nabla\phi_{\rm eik}|^2\right)\left(t,\widetilde{x}(t)\right) = -\nabla V\left(t,\widetilde{x}(t)\right).
\end{aligned}$$

We infer that $(\widetilde{x},\widetilde{\xi})$ solves (1.11). The lemma then follows from uniqueness for (1.11). □

Note that knowing $\nabla\phi_{\rm eik}$ suffices to get $\phi_{\rm eik}$ itself, which is given by

$$\phi_{\rm eik}(t,x) = \phi_0(x) - \int_0^t \left(\frac{1}{2}|\nabla\phi_{\rm eik}(\tau,x)|^2 + V(\tau,x)\right)d\tau.$$

The above lemma and the Local Inversion Theorem yield

Lemma 1.5. *Let V and ϕ_0 smooth as in Lemma 1.3. Let $t \in [-T,T]$ and θ_0 an open set of $\mathbb{R}^n$. Denote*

$$\theta_t := \{x(t,y) \in \mathbb{R}^n, y \in \theta_0\} \quad ; \quad \theta := \{(t,x) \in [-T,T]\times\mathbb{R}^n, x \in \theta_t\}.$$

Suppose that for $t \in [-T, T]$, the mapping

$$\theta_0 \ni y \mapsto x(t, y) \in \theta_t$$

is bijective, and denote by $y(t, x)$ its inverse. Assume also that

$$\nabla_x y \in L^\infty_{\rm loc}(\theta).$$

Then there exists a unique function $\theta \ni (t, x) \mapsto \phi_{\rm eik}(t, x) \in \mathbb{R}$ that solves (1.10)*, and satisfies $\nabla_x^2 \phi_{\rm eik} \in L^\infty_{\rm loc}(\theta)$. Moreover,*

$$\nabla \phi_{\rm eik}(t, x) = \xi(t, y(t, x)). \tag{1.13}$$

Note that the existence time T may depend on the neighborhood θ_0. It actually does in general, as shown by the following example.

Example 1.6. Assume that $V \equiv 0$ and

$$\phi_0(x) = -\frac{1}{(2+2\delta)T_c}\left(|x|^2+1\right)^{1+\delta}, \quad T_c > 0,\ \delta \geqslant 0.$$

For $\delta > 0$, integrating (1.11) yields:

$$\begin{aligned} x(t, y) &= y + \int_0^t \xi(s, y)ds = y + \int_0^t \xi(0, y)ds = y - \frac{t}{T_c}\left(|y|^2+1\right)^\delta y \\ &= y\left(1 - \frac{t}{T_c}\left(|y|^2+1\right)^\delta\right). \end{aligned}$$

For $R > 0$, we see that the rays starting from the ball $\{|y| = R\}$ meet at the origin at time

$$T_c(R) = \frac{T_c}{(R^2+1)^\delta}.$$

Since R is arbitrary, this shows that several rays can meet arbitrarily fast, thus showing that the above lemma cannot be applied uniformly in space.

Of course, the above issue would not appear if the space variable x belonged to a compact set instead of the whole space $\mathbb{R}^n$. To obtain a local time of existence with is independent of $y \in \mathbb{R}^n$, we have to make an extra assumption, in order to be able to apply a *global* inversion theorem.

Assumption 1.7 (Geometric assumption). *We assume that the potential and the initial phase are smooth, real-valued, and* subquadratic*:*

- *$V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$, and $\partial_x^\alpha V \in C(\mathbb{R}; L^\infty(\mathbb{R}^n))$ as soon as $|\alpha| \geqslant 2$.*
- *$\phi_0 \in C^\infty(\mathbb{R}^n)$, and $\partial^\alpha \phi_0 \in L^\infty(\mathbb{R}^n)$ as soon as $|\alpha| \geqslant 2$.*

As a consequence of this assumption on V, if $a_0^\varepsilon \in L^2$, then (1.9) has a unique solution $u^\varepsilon \in C(\mathbb{R}; L^2)$. See e.g. [Reed and Simon (1975)].

The following result can be found in [Schwartz (1969)], or in Appendix A of [Dereziński and Gérard (1997)].

Lemma 1.8. *Suppose that the function $\mathbb{R}^n \ni y \mapsto x(y) \in \mathbb{R}^n$ satisfies:*

$$|\det \nabla_y x| \geqslant C_0 > 0 \quad \textit{and} \quad \left|\partial_y^\alpha x\right| \leqslant C, \ |\alpha| = 1, 2.$$

Then x is bijective.

We can then prove

Proposition 1.9. *Under Assumption 1.7, there exists $T > 0$ and a unique solution $\phi_{\rm eik} \in C^\infty([-T,T] \times \mathbb{R}^n)$ to (1.10). In addition, this solution is subquadratic: $\partial_x^\alpha \phi_{\rm eik} \in L^\infty([-T,T] \times \mathbb{R}^n)$ as soon as $|\alpha| \geqslant 2$.*

Proof. We know that we can solve (1.11) locally in time in the neighborhood of any $y \in \mathbb{R}^n$. In order to apply the above global inversion result, differentiate (1.11) with respect to y:

$$\begin{cases} \partial_t \partial_y x(t,y) = \partial_y \xi\,(t,y) & ; \ \partial_y x(0,y) = \mathrm{Id}, \\ \partial_t \partial_y \xi(t,y) = -\nabla_x^2 V\,(t, x(t,y))\, \partial_y x(t,y) & ; \ \partial_y \xi(0,y) = \nabla^2 \phi_0(y). \end{cases} \tag{1.14}$$

Integrating (1.14) in time, we infer from Assumption 1.7 that for any $T > 0$, there exists C_T such that for $(t,y) \in [-T,T] \times \mathbb{R}^n$:

$$|\partial_y x(t,y)| + |\partial_y \xi(t,y)| \leqslant C_T + C_T \int_0^t \left(|\partial_y x(s,y)| + |\partial_y \xi(s,y)|\right) ds.$$

Gronwall lemma yields:

$$\|\partial_y x(t)\|_{L_y^\infty} + \|\partial_y \xi(t)\|_{L_y^\infty} \leqslant C'(T). \tag{1.15}$$

Similarly,

$$\left\|\partial_y^\alpha x(t)\right\|_{L_y^\infty} + \left\|\partial_y^\alpha \xi(t)\right\|_{L_y^\infty} \leqslant C(\alpha, T), \quad \forall \alpha \in \mathbb{N}^n, \ |\alpha| \geqslant 1. \tag{1.16}$$

Integrating the first line of (1.14) in time, we have:

$$\det \nabla_y x(t,y) = \det \left(\mathrm{Id} + \int_0^t \nabla_y \xi\,(s,y)\, ds\right).$$

We infer from (1.15) that for $t \in [-T,T]$, provided that $T > 0$ is sufficiently small, we can find $C_0 > 0$ such that:

$$|\det \nabla_y x(t,y)| \geqslant C_0, \quad \forall (t,y) \in [-T,T] \times \mathbb{R}^n. \tag{1.17}$$

Lemma 1.8 shows that we can invert $y \mapsto x(t,y)$ for $t \in [-T,T]$.

To apply Lemma 1.5 with $\theta_0 = \theta = \theta_t = \mathbb{R}^n$, we must check that $\nabla_x y \in L^\infty_{\rm loc}(\mathbb{R}^n)$. Differentiate the relation

$$x\left(t, y(t,x)\right) = x$$

with respect to x:

$$\nabla_x y(t,x) \nabla_y x\left(t, y(t,x)\right) = \mathrm{Id}.$$

Therefore, $\nabla_x y(t,x) = \nabla_y x\left(t, y(t,x)\right)^{-1}$ as matrices, and

$$\nabla_x y(t,x) = \frac{1}{\det \nabla_y x(t,y)} \mathrm{adj}\left(\nabla_y x\left(t, y(t,x)\right)\right), \tag{1.18}$$

where $\mathrm{adj}\left(\nabla_y x\right)$ denotes the adjugate of $\nabla_y x$. We infer from (1.15) and (1.17) that $\nabla_x y \in L^\infty(\mathbb{R}^n)$ for $t \in [-T,T]$. Therefore, Lemma 1.5 yields a smooth solution $\phi_{\rm eik}$ to (1.10); it is local in time and *global in space*: $\phi_{\rm eik} \in C^\infty([-T,T]\times\mathbb{R}^n)$.

The fact that $\phi_{\rm eik}$ is subquadratic as stated in Proposition 1.9 then stems from (1.13), (1.16), (1.17) and (1.18). □

Note that Example 1.6 shows that the above result is essentially sharp: if Assumption 1.7 is not satisfied, then the above result fails to be true. Similarly, if we consider $V = V(x) = -x^4$ in space dimension $n = 1$, then, also due to an infinite speed of propagation, the Hamiltonian $-\partial_x^2 - x^4$ is not essentially self-adjoint (see Chap. 13, Sect. 6, Cor. 22 in [Dunford and Schwartz (1963)]). We now give some examples of cases where the phase $\phi_{\rm eik}$ can be computed explicitly, which also show that in general, the above time T is necessarily finite.

Example 1.10 (Quadratic phase). *Resume Example 1.6, and consider the value $\delta = 0$. In that case, Assumption 1.7 is satisfied, and* (1.10) *is solved explicitly:*

$$\phi_{\rm eik}(t,x) = \frac{|x|^2}{2(t-T_c)} - \frac{1}{2T_c}.$$

This shows that we can solve (1.10) *globally in space, but only locally in time: as $t \to T_c$, $\phi_{\rm eik}$ ceases to be smooth. A* caustic *reduced to a single point (the origin) is formed.*

Remark 1.11. More generally, the space-time set where the map $y \mapsto x(t,y)$ ceases to be a diffeomorphism is called *caustic*. The behavior of the solution u^ε to (1.6) with $\lambda = 0$ is given for all time in terms of oscillatory integrals ([Duistermaat (1974); Maslov and Fedoriuk (1981)]). We present results concerning the asymptotic behavior of solutions to (1.6) with $\lambda \neq 0$ in the presence of point caustics in the second part of this book.

Example 1.12 (Harmonic potential). *When $\phi_0 \equiv 0$, and V is independent of time and quadratic, $V = V(x) = \frac{1}{2}\sum_{j=1}^n \omega_j^2 x_j^2$, we have:*

$$\phi_{\rm eik}(t,x) = -\sum_{j=1}^n \frac{\omega_j}{2} x_j^2 \tan(\omega_j t).$$

This also shows that we can solve (1.10) *globally in space, but locally in time only. Note that if we replace formally ω_j by $i\omega_j$, then V is turned into $-V$, and the trigonometric functions become hyperbolic functions: we can then solve* (1.10) *globally in space* and *time.*

Example 1.13 (Plane wave). *If we assume $V \equiv 0$ and $\phi_0(x) = \xi_0 \cdot x$ for some $\xi_0 \in \mathbb{R}^n$, then we find:*

$$\phi_{\rm eik}(t,x) = \xi_0 \cdot x - \frac{1}{2}|\xi_0|^2 t.$$

Also in this case, we can solve (1.10) *globally in space* and *time.*

1.3.2 *The transport equations*

To cancel the ε^1 term, the second step consists in solving (1.8):

$$\partial_t a + \nabla\phi_{\rm eik} \cdot \nabla a + \frac{1}{2} a \Delta \phi_{\rm eik} = 0 \quad ; \quad a(0,x) = a_0(x), \tag{1.19}$$

where a_0 is given as the first term in the asymptotic expansion of the initial amplitude (1.4). The equation is a transport equation (see e.g. [Evans (1998)]), since the characteristics for the operator $\partial_t + \nabla\phi_{\rm eik} \cdot \nabla$ do not meet for $t \in [-T,T]$, by construction. As a matter of fact, this equation can be solved rather explicitly, in terms of the geometric tools that we have used in the previous paragraph.

Introduce the *Jacobi's determinant*

$$J_t(y) = \det \nabla_y x(t,y),$$

where $x(t,y)$ is given by the Hamiltonian flow (1.11). Note that $J_0(y) = 1$ for all $y \in \mathbb{R}^n$. By construction, for $t \in [-T,T]$, the function $y \mapsto J_t(y)$ is uniformly bounded from above and from below:

$$\exists C > 0, \ \frac{1}{C} \leqslant J_t(y) \leqslant C, \quad \forall (t,y) \in [-T,T] \times \mathbb{R}^n.$$

Define the function A by

$$A(t,y) := a\left(t, x(t,y)\right) \sqrt{J_t(y)}.$$

Then since for $t \in [-T, T]$, $y \mapsto x(t, y)$ is a global diffeomorphism on $\mathbb{R}^n$, (1.19) is *equivalent* to the equation

$$\partial_t A(t, y) = 0 \quad ; \quad A(0, y) = a_0(y).$$

We obviously have $A(t, y) = a_0(y)$ for all $t \in [-T, T]$, and back to the function a, this yields

$$a(t, x) = \frac{1}{\sqrt{J_t(y(t, x))}} a_0\left(y(t, x)\right), \tag{1.20}$$

where $y(t, x)$ is the inverse map of $y \mapsto x(t, y)$.

Remark 1.14. The computations of Sec. 1.2 show that the amplitudes are given by

$$\partial_t a^{(j)} + \nabla\phi_{\text{eik}} \cdot \nabla a^{(j)} + \frac{1}{2} a^{(j)} \Delta\phi_{\text{eik}} = \frac{i}{2} \Delta a^{(j-1)} \quad ; \quad a^{(j)}_{|t=0} = a_j,$$

with the convention $a^{(-1)} = 0$ and $a^{(0)} = a$. For $j \geqslant 1$, this equation is the inhomogeneous analogue of (1.19). It can be solved by using the same change of variable as above. This shows that when ϕ_{eik} becomes singular (formation of a caustic), *all* the terms computed by this WKB analysis become singular in general. The WKB hierarchy ceases to be relevant at a caustic.

Proposition 1.15. *Let* $s \geqslant 0$ *and* $a_0 \in H^s(\mathbb{R}^n)$. *Then* (1.19) *has a unique solution* $a \in C([-T; T]; H^s)$, *where* $T > 0$ *is given by Proposition 1.9.*

Proof. Existence and uniqueness at the L^2 level stem from the above analysis, (1.20). To prove that an H^s regularity is propagated for $s > 0$, we could also use (1.20). We shall use another approach, which will be more natural in the nonlinear setting. To simplify the presentation, we assume $s \in \mathbb{N}$, and prove *a priori* estimates in H^s. Let $\alpha \in \mathbb{N}^n$, with $|\alpha| \leqslant s$. Applying ∂_x^α to (1.19), we find:

$$\partial_t \partial_x^\alpha a + \nabla\phi_{\text{eik}} \cdot \nabla \partial_x^\alpha a = \left[\nabla\phi_{\text{eik}} \cdot \nabla, \partial_x^\alpha\right] a - \frac{1}{2} \partial_x^\alpha \left(a \Delta\phi_{\text{eik}}\right) =: R_\alpha, \tag{1.21}$$

where $[P, Q] = PQ - QP$ denotes the commutator of the operators P and Q. Take the inner product of (1.21) with $\partial_x^\alpha a$, and consider the real part:

$$\frac{1}{2} \frac{d}{dt} \|\partial_x^\alpha a\|_{L^2}^2 + \operatorname{Re} \int_{\mathbb{R}^n} \overline{\partial_x^\alpha a} \nabla\phi_{\text{eik}} \cdot \nabla \partial_x^\alpha a \leqslant \|R_\alpha\|_{L^2} \|a\|_{H^s}.$$

Notice that we have

$$\begin{aligned} \left| \operatorname{Re} \int_{\mathbb{R}^n} \overline{\partial_x^\alpha a} \nabla\phi_{\text{eik}} \cdot \nabla \partial_x^\alpha a \right| &= \frac{1}{2} \left| \int_{\mathbb{R}^n} \nabla\phi_{\text{eik}} \cdot \nabla \left|\partial_x^\alpha a\right|^2 \right| \\ &= \frac{1}{2} \left| \int_{\mathbb{R}^n} \left|\partial_x^\alpha a\right|^2 \Delta\phi_{\text{eik}} \right| \leqslant C \|a\|_{H^s}^2, \end{aligned}$$

since $\Delta\phi_{\rm eik} \in L^\infty([-T,T]\times\mathbb{R}^n)$ from Proposition 1.9. Summing over α such that $|\alpha| \leqslant s$, we infer:

$$\frac{d}{dt}\|a\|_{H^s}^2 \leqslant C\left\|a\right\|_{H^s}^2 + \left\|R_\alpha\right\|_{H^s}^2 .$$

To apply Gronwall lemma, we need to estimate the last term: we use the fact that the derivatives of order at least two of $\phi_{\rm eik}$ are bounded, from Proposition 1.9, to have:

$$\left\|R_\alpha\right\|_{L^2} \leqslant C\left\|a\right\|_{H^s} .$$

We can then conclude:

$$\|a\|_{L^\infty([-T,T];H^s)} \leqslant C\left\|a_0\right\|_{H^s} ,$$

which completes the proof of the proposition. $\square$

Let us examine what can be deduced at this stage, and see which rigorous meaning can be given to the relation $u^\varepsilon \sim ae^{i\phi_{\rm eik}/\varepsilon}$. Let

$$v_1^\varepsilon(t,x) := a(t,x)e^{i\phi_{\rm eik}(t,x)/\varepsilon}.$$

Proposition 1.16. *Let $s \geqslant 2$, $a_0 \in H^s(\mathbb{R}^n)$, and Assumption 1.7 be satisfied. Suppose that*

$$\|a_0^\varepsilon - a_0\|_{H^{s-2}} = \mathcal{O}\left(\varepsilon^\beta\right)$$

for some $\beta > 0$. Then there exists $C > 0$ independent of $\varepsilon \in]0,1]$ such that

$$\sup_{t\in[-T,T]}\left\|u^\varepsilon(t) - v_1^\varepsilon(t)\right\|_{L^2} \leqslant C\varepsilon^{\min(1,\beta)},$$

where T is given by Proposition 1.9. If in addition $s > n/2+2$, then

$$\sup_{t\in[-T,T]}\left\|u^\varepsilon(t) - v_1^\varepsilon(t)\right\|_{L^\infty} \leqslant C\varepsilon^{\min(1,\beta)}.$$

Proof. Let $w_1^\varepsilon := u^\varepsilon - v_1^\varepsilon$. By construction, is solves

$$i\varepsilon\partial_t w_1^\varepsilon + \frac{\varepsilon^2}{2}\Delta w_1^\varepsilon = Vw_1^\varepsilon - \frac{\varepsilon^2}{2}e^{i\phi_{\rm eik}/\varepsilon}\Delta a \quad ; \quad w^\varepsilon_{1|t=0} = a_0^\varepsilon - a_0. \qquad (1.22)$$

By Lemma 1.2, which can be made rigorous in the present setting (exercise), we have:

$$\sup_{t\in[-T,T]}\|w_1^\varepsilon(t)\|_{L^2} \leqslant \|a_0^\varepsilon - a_0\|_{L^2} + \frac{\varepsilon}{2}\int_{-T}^T \|\Delta a(\tau)\|_{L^2}d\tau \leqslant C\left(\varepsilon^\beta + \varepsilon\right),$$

where we have used the assumption on $a_0^\varepsilon - a_0$ and Proposition 1.15. This yields the first estimate of the proposition.

To prove the second estimate, we want to use the Sobolev embedding $H^s \subset L^\infty$ for $s > n/2$. A first idea could be to differentiate (1.22) with respect to space variables, and use Lemma 1.2. However, this direct approach fails, because the source term

$$\frac{\varepsilon^2}{2} e^{i\phi_{\rm eik}/\varepsilon} \Delta a$$

is of order $\mathcal{O}(\varepsilon^2)$ in L^2, but of order $\mathcal{O}(\varepsilon^{2-s})$ in H^s, $s \geqslant 0$. This is due to the rapidly oscillatory factor $e^{i\phi_{\rm eik}/\varepsilon}$. Moreover, under our assumptions, it is not guaranteed that $\nabla\phi_{\rm eik}\Delta a \in C([-T,T];L^2)$, since $\nabla\phi_{\rm eik}$ may grow linearly with respect to the space variable, as shown by Examples 1.10 and 1.12. We therefore adopt a different point of view, relying on the remark:

$$|u^\varepsilon - v_1^\varepsilon| = \left|u^\varepsilon - ae^{i\phi_{\rm eik}/\varepsilon}\right| = \left|u^\varepsilon e^{-i\phi_{\rm eik}/\varepsilon} - a\right|.$$

Set $a^\varepsilon := u^\varepsilon e^{-i\phi_{\rm eik}/\varepsilon}$. We check that it solves

$$\partial_t a^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi_{\rm eik} = i\frac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon_{|t=0} = a_0^\varepsilon.$$

Therefore, $r^\varepsilon = a^\varepsilon - a = w_1^\varepsilon e^{-i\phi_{\rm eik}/\varepsilon}$ solves

$$\partial_t r^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla r^\varepsilon + \frac{1}{2}r^\varepsilon\Delta\phi_{\rm eik} = i\frac{\varepsilon}{2}\Delta r^\varepsilon + i\frac{\varepsilon}{2}\Delta a \quad ; \quad r^\varepsilon_{|t=0} = a_0^\varepsilon - a_0. \tag{1.23}$$

Note that this equation is very similar to the transport equation (1.19), with two differences. First, the presence of the operator $i\varepsilon\Delta$ acting on r^ε on the right hand side. Second, the source term $i\varepsilon\Delta a$, which makes the equation inhomogeneous.

We know by construction that $r^\varepsilon \in C([-T,T];L^2)$, and we seek *a priori* estimates in $C([-T,T];H^k)$. These are established along the same lines as in the proof of Proposition 1.15. We note that since the operator $i\Delta$ is skew-symmetric on H^s, the term $i\varepsilon\Delta r^\varepsilon$ vanishes from the energy estimates in H^s. Then, the source term is of order ε in $C([-T;T];H^{s-2})$ from Proposition 1.15. We infer:

$$\sup_{t\in[-T,T]} \|r^\varepsilon(t)\|_{H^{s-2}} \lesssim \varepsilon^\beta + \varepsilon.$$

Note that this estimate, along with a standard continuation argument, shows that $a^\varepsilon \in C([-T;T];H^{s-2})$ for $\varepsilon > 0$ sufficiently small. Since $s-2 > n/2$, we deduce

$$\sup_{t\in[-T,T]} \|r^\varepsilon(t)\|_{L^\infty} \lesssim \varepsilon^\beta + \varepsilon,$$

which completes the proof of the proposition. $\square$

Before analyzing the accuracy of higher order approximate solutions, let us examine the candidate v_1^ε in the case of the examples given in Sec. 1.3.1.

Example 1.17 (Quadratic phase). *Resume Example 1.10. In this case, we compute, for $t < T_c$,*

$$a(t,x) = \left(\frac{T_c}{T_c - t}\right)^{n/2} a_0\left(\frac{T_c}{T_c - t}x\right).$$

As $t \to T_c$, not only $\phi_{\rm eik}$ ceases to be smooth, but also a. This is a general feature of the formation of caustics: all the terms constructed by the usual WKB analysis become singular.

Example 1.18 (Harmonic potential). *Resume Example 1.12. If $|t|$ is sufficiently small so that $\phi_{\rm eik}$ remains smooth on $[0,t]$, we find:*

$$a(t,x) = \prod_{j=1}^n \left(\frac{1}{\cos(\omega_j t)}\right)^{1/2} a_0\left(\frac{x_1}{\cos(\omega_1 t)}, \dots, \frac{x_n}{\cos(\omega_n t)}\right).$$

Here again, $\phi_{\rm eik}$ and a become singular simultaneously.

Example 1.19 (Plane wave). *If we assume $V \equiv 0$ and $\phi_0(x) = \xi_0 \cdot x$ for some $\xi_0 \in \mathbb{R}^n$, then we find:*

$$a(t,x) = a_0\left(x - \xi_0 t\right).$$

The initial amplitude is simply transported with constant velocity.

We can continue this analysis to arbitrary order:

Proposition 1.20. *Let $k \in \mathbb{N}\setminus\{0\}$ and $s \geqslant 2k+2$. Let $a_0, a_1, \dots, a_k$ with $a_j \in H^{s-2j}(\mathbb{R}^n)$, and let Assumption 1.7 be satisfied. Suppose that*

$$\|a_0^\varepsilon - a_0 - \varepsilon a_1 - \ldots - \varepsilon^k a_k\|_{H^{s-2k-2}} = \mathcal{O}\left(\varepsilon^{k+\beta}\right)$$

for some $\beta > 0$. Then we can find $a^{(1)}, \dots, a^{(k)}$, with

$$a^{(j)} \in C([-T,T]; H^{s-2j}),$$

such that if we set

$$v_{k+1}^\varepsilon = \left(a + \varepsilon a^{(1)} + \ldots + \varepsilon^k a^{(k)}\right) e^{i\phi_{\rm eik}/\varepsilon},$$

there exists $C > 0$ independent of $\varepsilon \in]0,1]$ such that

$$\sup_{t\in[-T,T]} \left\|\left(u^\varepsilon(t) - v_{k+1}^\varepsilon(t)\right) e^{-i\phi_{\rm eik}(t)/\varepsilon}\right\|_{H^{s-2k-2}} \leqslant C\varepsilon^{\min(k+1,k+\beta)},$$

where T is given by Proposition 1.9.

Proof. We simply sketch the proof, since it follows arguments which have been introduced above. First, the computations presented in Sec. 1.2 show that to cancel the term in ε^{j+1}, $1 \leqslant j \leqslant k$, we naturally impose:

$$\partial_t a^{(j)} + \nabla\phi_{\rm eik} \cdot \nabla a^{(j)} + \frac{1}{2} a^{(j)} \Delta\phi_{\rm eik} = \frac{i}{2}\Delta a^{(j-1)} \quad ; \quad a^{(j)}_{|t=0} = a_j.$$

This equation is the inhomogeneous analogue of (1.19). Using the same arguments as in the proof of Proposition 1.15, it is easy to see that it has a unique solution $a^{(j)} \in C([-T,T];L^2)$, whose spatial regularity is that of $a^{(j-1)}$, minus 2. Starting an induction with Proposition 1.15, we construct

$$a^{(j)} \in C([-T,T];H^{s-2j}).$$

To prove the error estimate, introduce $r_k^\varepsilon = a^\varepsilon - a - \varepsilon a^{(1)} - \ldots - \varepsilon^k a^{(k)}$, where we recall that $a^\varepsilon = u^\varepsilon e^{-i\phi_{\rm eik}/\varepsilon}$. By construction, the remainder r_k^ε is in $C([-T;T];L^2)$ since $s \geqslant 2k+2$, and it solves:

$$\left\{\begin{aligned} \partial_t r_k^\varepsilon + \nabla\phi_{\rm eik} \cdot \nabla r_k^\varepsilon + \frac{1}{2} r_k^\varepsilon \Delta\phi_{\rm eik} = i\frac{\varepsilon}{2}\Delta r_k^\varepsilon + i\frac{\varepsilon^{k+1}}{2}\Delta a^{(k)},\\ r^\varepsilon_{|t=0} = a_0^\varepsilon - a_0 - \ldots - \varepsilon^k a_k. \end{aligned}\right.$$

We can then mimic the end of the proof of Proposition 1.16. $\square$

To conclude, we see that we can construct an arbitrarily accurate (as $\varepsilon \to 0$) approximation of u^ε on $[-T,T]$, provided that the initial profiles a_j are sufficiently smooth. The goal now is to see how this approach can be adapted to a nonlinear framework.

1.4 Basic results in the nonlinear case

Before presenting a WKB analysis in the case $f \neq 0$ in (1.1), we recall a few important facts about the nonlinear Cauchy problem for (1.1). We shall simply gather classical results, which can be found for instance in [Cazenave and Haraux (1998); Cazenave (2003); Ginibre and Velo (1985a); Kato (1989); Tao (2006)]. Several notions of solutions are available. According to the cases, we will work with the notion of strong solutions (Chapters 2, 4 and 5), of weak solutions (Chapters 3 and 5) or of mild solutions (especially in the second part of this book).

In this section, one should think that the parameter $\varepsilon > 0$ is *fixed*. The dependence upon ε is discussed in the forthcoming sections.

1.4.1 *Formal properties*

Since V and f are real-valued, the L^2 norm of u^ε is formally independent of time:

$$\|u^\varepsilon(t)\|_{L^2} = \|u^\varepsilon(0)\|_{L^2}. \tag{1.24}$$

This can be seen from the proof of Lemma 1.2, with $F^\varepsilon = V + f\left(|u^\varepsilon|^2\right)$ and $R^\varepsilon = 0$. This relation yields an *a priori* bound for the L^2 norm of u^ε.

When the potential V is time-independent, $V = V(x)$, (1.1) has a Hamiltonian structure. Introduce

$$F(y) = \int_0^y f(\eta)d\eta.$$

The following energy is formally independent of time:

$$\begin{aligned} E^\varepsilon(u^\varepsilon(t)) =& \frac{1}{2}\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2}^2 + \int_{\mathbb{R}^n} F\left(|u^\varepsilon(t,x)|^2\right)dx \\ &+ \int_{\mathbb{R}^n} V(x)|u^\varepsilon(t,x)|^2 dx. \end{aligned} \tag{1.25}$$

We see that if E^ε is finite, and if $V \geqslant 0$ and $F \geqslant 0$, then this yields an *a priori* bound on $\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2}$.

Example 1.21. If $V = V(x) \geqslant$ and $f(y) = \lambda y^\sigma$, then (1.25) becomes

$$E^\varepsilon = \frac{1}{2}\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\int_{\mathbb{R}^n} |u^\varepsilon(t,x)|^{2\sigma+2}dx + \int_{\mathbb{R}^n} V(x)|u^\varepsilon(t,x)|^2 dx.$$

If $\lambda \geqslant 0$ (*defocusing nonlinearity*), this yields an *a priori* bound on $\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2}$. On the other hand, if $\lambda < 0$, then $\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2}$ may become unbounded in finite time: this is the *finite time blow-up* phenomenon (see e.g. [Cazenave (2003); Sulem and Sulem (1999)]). Since the L^2 norm of u^ε is conserved, one can replace the assumption $V \geqslant 0$ with $V \geqslant -C$ for some $C > 0$, and leave the above discussion unchanged.

Example 1.22. If V is unbounded from below, the conservation of the energy does not seem to provide interesting informations. For instance, if $V(x) = -|x|^2$, then even in the linear case $f = 0$, the energy is not a positive energy functional (see [Carles (2003a)] though, for the nonlinear Cauchy problem).

1.4.2 *Strong solutions*

A remarkable fact is that if the external potential V is subquadratic in the sense of Assumption 1.7, then one can define a strongly continuous semi-group for the linear equation (1.9). As we have mentioned already, if no sign assumption is made on V, then Assumption 1.7 is essentially sharp: if $n = 1$ and $V(x) = -x^4$, then $-\partial_x^2 + V$ is not essentially self-adjoint on the set of test functions ([Dunford and Schwartz (1963)]). Under Assumption 1.7, one defines $U^\varepsilon(t,s)$ such that $u^\varepsilon_{\rm lin}(t,x) = U^\varepsilon(t,s)\varphi^\varepsilon(x)$, where

$$i\varepsilon\partial_t u^\varepsilon_{\rm lin} + \frac{\varepsilon^2}{2}\Delta u^\varepsilon_{\rm lin} = V u^\varepsilon_{\rm lin} \quad ; \quad u^\varepsilon_{\rm lin}(s,x) = \varphi^\varepsilon(x).$$

Note that $U^\varepsilon(t,t) = \mathrm{Id}$. The existence of $U^\varepsilon(t,s)$ is established in [Fujiwara (1979)], along with the following properties:

- The map $(t,s) \mapsto U^\varepsilon(t,s)$ is strongly continuous.
- $U^\varepsilon(t,s)^* = U^\varepsilon(t,s)^{-1}$.
- $U^\varepsilon(t,\tau)U^\varepsilon(\tau,s) = U^\varepsilon(t,s)$.
- $U^\varepsilon(t,s)$ is unitary on L^2: $\|U^\varepsilon(t,s)\varphi^\varepsilon\|_{L^2} = \|\varphi^\varepsilon\|_{L^2}$.

We construct strong solutions which are (at least) in $H^s(\mathbb{R}^n)$, for $s > n/2$. Recall that H^s is then an algebra, embedded into $L^\infty(\mathbb{R}^n)$. We shall also use the following version of Schauder's lemma:

Lemma 1.23 (Schauder's lemma). *Suppose that $G : \mathbb{C} \to \mathbb{C}$ is a smooth function, such that $G(0) = 0$. Then the map $u \mapsto G(u)$ sends $H^s(\mathbb{R}^n)$ to itself provided $s > n/2$. The map is uniformly Lipschitzean on bounded subsets of H^s.*

We refer to [Taylor (1997)] or [Rauch and Keel (1999)] for the proof of this result, as well as to the following refinement (*tame estimate*):

Lemma 1.24 (Moser's inequality). *Suppose that $G : \mathbb{C} \to \mathbb{C}$ is a smooth function, such that $G(0) = 0$. Then there exists $C : [0,\infty[\to [0,\infty[$ such that for all $u \in H^s(\mathbb{R}^n)$,*

$$\|G(u)\|_{H^s} \leqslant C\left(\|u\|_{L^\infty}\right)\|u\|_{H^s}.$$

For $k \in \mathbb{N}$, denote

$$\Sigma(k) = H^k \cap \mathcal{F}(H^k) = \{f \in H^k(\mathbb{R}^n)\ ;\ x \mapsto \langle x\rangle^k f(x) \in L^2(\mathbb{R}^n)\}.$$

Proposition 1.25. *Let V satisfy Assumption 1.7, and let $f \in C^\infty(\mathbb{R}_+;\mathbb{R})$. Let $k \in \mathbb{N}$, with $k > n/2$, and fix $\varepsilon \in]0,1]$.*

- *If $u_0^\varepsilon \in \Sigma(k)$, then there exist $T_-^\varepsilon, T_+^\varepsilon > 0$ and a unique maximal solution $u^\varepsilon \in C(]-T_-^\varepsilon, T_+^\varepsilon[;\Sigma(k))$ to (1.1), such that $u^\varepsilon_{|t=0} = u_0^\varepsilon$. It is maximal in the sense that if, say, $T_+^\varepsilon < \infty$, then*

$$\limsup_{t\to T_+^\varepsilon} \|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^n)} = +\infty. \tag{1.26}$$

- *Assume in addition that V is sub-linear: $\nabla V \in L^\infty_{\rm loc}(\mathbb{R}; L^\infty(\mathbb{R}^n))$. Let $s > n/2$ (not necessarily an integer). If $u_0^\varepsilon \in H^s(\mathbb{R}^n)$, then there exist $T_-^\varepsilon, T_+^\varepsilon > 0$ and a unique maximal solution $u^\varepsilon \in C(]-T_-^\varepsilon, T_+^\varepsilon[;H^s)$ to (1.1), such that $u^\varepsilon_{|t=0} = u_0^\varepsilon$. It is maximal in the sense that if, say, $T_+^\varepsilon < \infty$, then (1.26) holds. In particular, if $u_0^\varepsilon \in H^\infty$, then $u^\varepsilon \in C^\infty(]-T_-^\varepsilon, T_+^\varepsilon[;H^\infty)$.*

Proof. The proof follows arguments which are classical in the context of semilinear evolution equations. We indicate a few important facts, and refer to [Cazenave and Haraux (1998)] to fill the gaps.

The general idea consists in applying a fixed point argument on the Duhamel's formulation of (1.1) with associated initial datum u_0^ε:

$$u^\varepsilon(t) = U^\varepsilon(t,0)u_0^\varepsilon - i\varepsilon^{-1}\int_0^t U^\varepsilon(t,\tau)\left(f\left(|u^\varepsilon(\tau)|^2\right)u^\varepsilon(\tau)\right)d\tau. \tag{1.27}$$

We claim that for any $k \in \mathbb{N}$ and any $T > 0$,

$$\sup_{t\in[-T,T]} \|U^\varepsilon(t,0)u_0^\varepsilon\|_{\Sigma(k)} \leqslant C(k,T)\|u_0^\varepsilon\|_{\Sigma(k)}. \tag{1.28}$$

For $k = 0$, this is due to the fact that $U^\varepsilon(t,0)$ is unitary on $L^2(\mathbb{R}^n)$. For $k = 1$, notice the commutator identities

$$\left[\nabla, i\varepsilon\partial_t + \frac{\varepsilon^2}{2}\Delta - V\right] = -\nabla V \quad ; \quad \left[x, i\varepsilon\partial_t + \frac{\varepsilon^2}{2}\Delta - V\right] = -\varepsilon^2\nabla. \tag{1.29}$$

By Assumption 1.7, $|\nabla V(t,x)| \leqslant C(T)\langle x\rangle$ for $|t| \leqslant T$, and (1.28) follows for $k = 1$. For $k \geqslant 2$, the proof follows the same lines.

To estimate the nonlinear term, we can assume without loss of generality that $f(0) = 0$. Indeed, we can replace f with $f - f(0)$ and V with $V + f(0)$. Schauder's lemma shows that

$$u \mapsto f\left(|u|^2\right)u$$

sends $H^s(\mathbb{R}^n)$ (resp. $\Sigma(k)$) to itself, provided $s > n/2$ (resp. $k > n/2$), and the map is uniformly Lipschitzean on bounded subsets of $H^s(\mathbb{R}^n)$ (resp. $\Sigma(k)$). The existence and the uniqueness of a solution in the first part of the proposition follow easily. The notion of maximality is then a consequence of Lemma 1.24.

When V is sub-linear, notice that in view of the commutator identities (1.29), the estimate (1.28) can be replaced with

$$\sup_{t\in[-T,T]} \|U^\varepsilon(t,0)u_0^\varepsilon\|_{H^s} \leqslant C(s,T)\|u_0^\varepsilon\|_{H^s}.$$

This is straightforward if $s \in \mathbb{N}$, and follows by interpolation for general $s \geqslant 0$. The proof of the second part of the proposition then follows the same lines as the first part. Finally, if $u_0^\varepsilon \in H^\infty$, then u^ε is also smooth with respect to the time variable, $u^\varepsilon \in C^\infty(]-T_-^\varepsilon, T_+^\varepsilon[; H^\infty)$, by a bootstrap argument. □

Note that the times T_-^ε and T_+^ε may very well go to zero as $\varepsilon \to 0$. The fact that we can bound these two quantities by $T > 0$ independent of $\varepsilon \in]0,1]$ is also a non-trivial information which will be provided by WKB analysis.

1.4.3 *Mild solutions*

Until the end of Sec. 1.4, to simplify the notations, we assume that the nonlinearity is homogeneous:

$$f(y) = \lambda y^\sigma, \quad \lambda \in \mathbb{R}, \sigma > 0.$$

In view of the conservations of mass (1.24) and energy (1.25), it is natural to look for solutions to (1.1) with initial data which are not necessarily as smooth as in Proposition 1.25. Typically, rather that (1.1), we rather consider its Duhamel's formulation, which now reads

$$u^\varepsilon(t) = U^\varepsilon(t,0)u_0^\varepsilon - i\lambda\varepsilon^{-1}\int_0^t U^\varepsilon(t,\tau)\left(|u^\varepsilon(\tau)|^{2\sigma}u^\varepsilon(\tau)\right)d\tau. \tag{1.30}$$

An extra property of U^ε was proved in [Fujiwara (1979)], which becomes interesting at this stage, that is, a dispersive estimate:

$$\|U^\varepsilon(t,0)U^\varepsilon(s,0)^*\varphi\|_{L^\infty(\mathbb{R}^n)} = \|U^\varepsilon(t,s)\varphi\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{C}{\left(\varepsilon|t-s|\right)^{n/2}}\|\varphi\|_{L^1(\mathbb{R}^n)},$$

provided that $|t-s| \leqslant \delta$, where C and $\delta > 0$ are independent of $\varepsilon \in]0,1]$. As a consequence, Strichartz estimates are available for U^ε (see e.g. [Keel and Tao (1998)]). Note that as $\varepsilon \to 0$, this dispersion estimate becomes worse and worse: the semi-classical limit $\varepsilon \to 0$ is sometimes referred to as *dispersionless limit.* Denoting

$$p = \frac{4\sigma+4}{n\sigma},$$

(the pair $(p, 2\sigma+2)$ is *admissible*, see Definition 7.4), we infer:

Proposition 1.26. *Let V satisfying Assumption 1.7.*
• *If $\sigma < 2/n$ and $u_0^\varepsilon \in L^2$, then* (1.30) *has a unique solution*

$$u^\varepsilon \in C(\mathbb{R}; L^2) \cap L^p_{\rm loc}(\mathbb{R}; L^{2\sigma+2}),$$

and (1.24) *holds for all $t \in \mathbb{R}$.*
• *If $u_0^\varepsilon \in \Sigma(1)$ and $\sigma < 2/(n-2)$ when $n \geqslant 3$, then there exist $T_-^\varepsilon, T_+^\varepsilon > 0$ and a unique solution*

$$u^\varepsilon \in C(]-T_-^\varepsilon, T_+^\varepsilon[; \Sigma(1)) \cap L^p_{\rm loc}(]-T_-^\varepsilon, T_+^\varepsilon[; W^{1,2\sigma+2})$$

to (1.30)*. Moreover, the mass* (1.24) *and the energy* (1.25) *do not depend on $t \in]-T_-^\varepsilon, T_+^\varepsilon[$.*
• *If $V = V(x)$ is sub-linear, $u_0^\varepsilon \in H^1$ and $\sigma < 2/(n-2)$ when $n \geqslant 3$, then there exist $T_-^\varepsilon, T_+^\varepsilon > 0$ and a unique solution*

$$u^\varepsilon \in C(]-T_-^\varepsilon, T_+^\varepsilon[; H^1) \cap L^p_{\rm loc}(]-T_-^\varepsilon, T_+^\varepsilon[; W^{1,2\sigma+2})$$

to (1.30)*. Moreover, the mass* (1.24) *does not depend on $t \in]-T_-^\varepsilon, T_+^\varepsilon[$. If the energy* (1.25) *is finite at time $t = 0$, then it is independent of $t \in]-T_-^\varepsilon, T_+^\varepsilon[$. If $\lambda \geqslant 0$, then we can take $T_-^\varepsilon = T_+^\varepsilon = \infty$, even if the energy is infinite.*
• *If $V = 0$, $u_0^\varepsilon \in \Sigma(1)$ and $\sigma < 2/(n-2)$ when $n \geqslant 3$, then the following evolution law holds so long as $u^\varepsilon \in C_t\Sigma(1)$:*

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\left\|(x+i\varepsilon t\nabla)\, u^\varepsilon\right\|_{L^2}^2 + \frac{\lambda t^2}{\sigma+1}\|u^\varepsilon\|_{L^{2\sigma+2}}^{2\sigma+2}\right) &= \\ = \frac{\lambda t}{\sigma+1}(2-n\sigma)\|u^\varepsilon\|_{L^{2\sigma+2}}^{2\sigma+2}&.\end{aligned} \tag{1.31}$$

In particular, if $\lambda \geqslant 0$, then $T_-^\varepsilon = T_+^\varepsilon = \infty$, and $u^\varepsilon \in C(\mathbb{R}; \Sigma(1))$.

Proof. The first point follows from the result of Y. Tsutsumi in the case $V = 0$ [Tsutsumi (1987)]. The proof relies on Strichartz estimates. The case $V \neq 0$ proceeds along the same lines, since local in time Strichartz estimates are available thanks to Assumption 1.7: the local in time result is made global thanks to the conservation of mass (1.24), since the local existence time depends only on the L^2 norm of the initial data.

The second point can be found in [Cazenave (2003)] in the case $V = 0$. To adapt it to the case $V \neq 0$, notice that (1.29) show that a closed family of estimates is available for u^ε, ∇u^ε and xu^ε. It is then possible to mimic the proof of the case $V = 0$. For the conservations of mass and energy, we refer to [Cazenave (2003)].

When V is sub-linear, it is possible to work in H^1 only, since

$$\left[\nabla, i\varepsilon\partial_t + \frac{\varepsilon^2}{2}\Delta - V\right] = -\nabla V$$

belongs to $L^\infty_{\rm loc}(\mathbb{R}; L^\infty(\mathbb{R}^n))$. For the global existence result, rewrite formally the conservation of the energy as

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|\varepsilon\nabla u^\varepsilon(t)\|^2_{L^2} + \frac{\lambda}{\sigma+1}\|u^\varepsilon(t)\|^{2\sigma+2}_{L^{2\sigma+2}}\right) &= -\frac{d}{dt}\int_{\mathbb{R}^n} V(x)|u^\varepsilon(t,x)|^2 dx \\ = -2\,\mathrm{Re}\int_{\mathbb{R}^n} V(x)\overline{u}^\varepsilon\partial_t u^\varepsilon dx &= -2\,\mathrm{Im}\int_{\mathbb{R}^n} V(x)\overline{u}^\varepsilon\left(i\partial_t u^\varepsilon\right)dx \\ = \mathrm{Im}\int_{\mathbb{R}^n} V(x)\overline{u}^\varepsilon\varepsilon\Delta u^\varepsilon dx &= -\,\mathrm{Im}\int_{\mathbb{R}^n}\overline{u}^\varepsilon\nabla V(x)\cdot\varepsilon\nabla u^\varepsilon dx.\end{aligned}$$

We conclude thanks to Cauchy–Schwarz inequality, the conservation of mass and Gronwall lemma, that $\|\nabla u^\varepsilon(t)\|_{L^2}$ remains bounded on bounded time intervals. Therefore the solution is global in time. See [Carles (2008)] for details.

The identity of the last point follows from the *pseudo-conformal conservation law*, derived by J. Ginibre and G. Velo [Ginibre and Velo (1979)] for $\varepsilon = 1$. The case $\varepsilon \in]0,1]$ is easily inferred *via* the scaling

$$(t,x) \mapsto \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

Since from the previous point, $\varepsilon\nabla u^\varepsilon \in C(\mathbb{R}; L^2)$ and $u^\varepsilon \in C(\mathbb{R}; L^{2\sigma+2})$, this evolution law shows the *a priori* estimate $xu^\varepsilon \in L^\infty_{\rm loc}\left(\mathbb{R}; L^2\right)$. □

1.4.4 *Weak solutions*

We will mention weak solutions only in the case $V = 0$, for a defocusing power-like nonlinearity. We therefore consider

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^{2\sigma}u^\varepsilon \quad ; \quad u^\varepsilon_{|t=0} = u^\varepsilon_0. \tag{1.32}$$

Definition 1.27 (Weak solution). *Let $u^\varepsilon_0 \in H^1 \cap L^{2\sigma+2}(\mathbb{R}^n)$. A (global) weak solution to* (1.32) *is a function $u^\varepsilon \in C(\mathbb{R}; \mathcal{D}') \cap L^\infty(\mathbb{R}; H^1 \cap L^{2\sigma+2})$ solving* (1.32) *in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}^n) \cap C(\mathbb{R}; L^2)$, and such that:*

- $\|u^\varepsilon(t)\|_{L^2} = \|u^\varepsilon_0\|_{L^2}$, $\forall t \in \mathbb{R}$.
- $E^\varepsilon(u^\varepsilon(t)) \leqslant E^\varepsilon(u^\varepsilon_0)$, $\forall t \in \mathbb{R}$.

Essentially, the energy conservation is replaced by an inequality, due to a limiting procedure and the use of Fatou's lemma in the construction of weak solutions.

Proposition 1.28 ([Ginibre and Velo (1985a)]). *Let $\sigma > 0$, $\varepsilon \in]0,1]$, and $u_0^\varepsilon \in H^1 \cap L^{2\sigma+2}(\mathbb{R}^n)$. Then (1.32) has a global weak solution. Moreover, if $\sigma < 2/(n-2)$, then this weak solution is unique, and coincides with the mild solution of the last point in Proposition 1.26.*