Loss of regularity for super-critical nonlinear Schrödinger equations

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(joint work with Thomas Alazard)

We consider the nonlinear Schrödinger equation with defocusing, smooth, nonlinearity:

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = |\psi|^{2\sigma}\psi, \quad \sigma \in \mathbb{N},\ x \in \mathbb{R}^n.$$

The critical index given by scaling arguments is

$$s_c = \frac{n}{2} - \frac{1}{\sigma}.$$

We assume $s_c > 0$ (the nonlinearity is L^2 super-critical). If $\psi_{|t=0} \in H^s$ with $0 < s < s_c$, it is known that the Cauchy problem is ill-posed in H^s [5]. We show that this is even worse: there is a loss of regularity (in any space dimension). A consequence of this result is easy to state for energy super-critical problems: assume $n \geqslant 3$ and $\sigma > 2/(n-2)$. We can find a sequence of initial data $(\varphi^\lambda)_{0<\lambda\leqslant 1}$ in the Schwartz class, and a sequence of time $t^\lambda \to 0$, such that the mass and the nonlinear energy of φ^λ go to zero as $\lambda \to 0$, and

$$\left\|\psi^\lambda\left(t^\lambda\right)\right\|_{H^s} \to +\infty \text{ as } \lambda \to 0, \quad \forall s > 1,$$

where ψ^λ is the solution to the nonlinear Schrödinger equation with data φ^λ. Since for strong solutions, the energy is conserved, and for weak solutions, it is at most the initial energy, this result is sharp. This result is in the same spirit of the pioneering work of G. Lebeau [6] for the wave equation. However, it seems that the method of G. Lebeau does not work so nicely in the case of Schrödinger equation; our proof follows a different approach, which is inspired by WKB analysis and fluid mechanics. This both simplifies and generalizes the proof in [4], which treated only the case $\sigma = 1$.

The proof proceeds in three steps. First, we reduce the problem to the study of the nonlinear Schrödinger equation in a high frequency régime:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^{2\sigma} u^\varepsilon \quad ; \quad u^\varepsilon(0,x) = a_0(x),$$

where a_0 is any non-trivial function in the Schwartz class, independent of the semi-classical parameter ε. The main result then follows from the fact that for t of order 1 (as $\varepsilon \to 0$), u^ε is exactly ε-oscillatory. The rest of the analysis consists in establishing this fact.

Second, we consider the expected limiting system: seeking $u^\varepsilon \approx ae^{i\phi/\varepsilon}$, let

$$(1) \qquad \begin{cases} \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + |a|^{2\sigma} = 0 \quad ; \qquad \phi_{|t=0} = 0. \\ \partial_t a + \nabla\phi\cdot\nabla a + \frac{1}{2}a\Delta\phi = 0 \quad ; \quad a_{|t=0} = a_0. \end{cases}$$

In terms of $(\nabla\phi, |a|^2)$, it is a compressible, isentropic Euler equation. Because of the possible presence of vacuum, this problem is not directly hyperbolic. Using an intermediate nonlinear change of unknown function due to T. Makino, S. Ukai and S. Kawashima [7], we show that this system is well-posed in Sobolev space, with a loss of at most one derivative.

The last step consists in proving a mild convergence of u^ε to the Euler type system, using a modulated energy functional *à la* Y. Brenier [3]. By mild convergence, we mean that we do not need to describe the asymptotic of u^ε in L^2 (for instance). Essentially, we need to know the behavior of $|u^\varepsilon|$ and $|\nabla u^\varepsilon|$ only:

$$\|(\varepsilon\nabla - i\nabla\phi)u^\varepsilon\|^2_{L^\infty([0,T];L^2)} + \left\| \left(|u^\varepsilon|^2 - |a|^2\right)^2 \left(|u^\varepsilon|^{2\sigma-2} + |a|^{2\sigma-2}\right)\right\|_{L^\infty([0,T];L^1)} = \mathcal{O}(\varepsilon^2).$$

Using Hölder's inequality, we give a rigorous meaning to the approximations:

$$\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2} \approx \|u^\varepsilon(t)\nabla\phi(t)\|_{L^2} \approx \|a(t)\nabla\phi(t)\|_{L^2}.$$

Using small time properties of the solution to (1), we see that there exists $\tau > 0$ independent of ε such that the last term is positive at time $t = \tau$. This shows that u^ε has become ε-oscillatory at time τ, hence the result.

Note that the study of (1) does not suffice to infer the limiting behavior of the wave function itself, due to more subtle modulation phenomena:

$$\left\| u^\varepsilon(t) - a(t)e^{i\phi^{(1)}(t)}e^{i\phi(t)/\varepsilon}\right\|_{L^2} \to 0 \text{ as } \varepsilon \to 0, \text{ for } t = \mathcal{O}(1),$$

where $\phi^{(1)}(t,x) = \mathcal{O}(t)$ is L^∞, and is non-trivial in general. Also, it is not possible to prove the above mentioned convergence by applying the Gronwall lemma for Schrödinger equations. In view of this aspect, the proof proposed to show the loss of regularity phenomenon is rather cheap: we establish the minimal information needed to conclude (we do not need to consider $\phi^{(1)}$). See [1] for the proof. Note also that this proof allows to consider weak solutions of the nonlinear Schrödinger equation, even if we have proved in a subsequent work [2] that, at least when $n \leqslant 3$, the solution u^ε remains a strong solution on the time interval that we consider.

References

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