Semi-classical analysis for the nonlinear Schrödinger equation with potential

Rémi Carles[1,*]

[1] CNRS & Université Montpellier 2, Mathématiques, CC 051, Place Eugène Bataillon, F-34095 Montpellier cedex 5, France

We describe the semi-classical limit for solutions of the defocusing nonlinear Schrödinger equation in the presence of an external potential in several régimes.

1 Introduction

The nonlinear Schrödinger equation is usually considered as a basic model to describe Bose–Einstein Condensation (see *e.g.* [1, 2]). The confining magnetic trap may be modeled by an external potential, *e.g.* an harmonic potential. In the most common model of a cubic nonlinearity, assuming that the interaction is repulsive (*e.g.* ^{87}Rb, ^{23}Na and ^{1}H), this leads to

$$i\varepsilon\partial_t \mathbf{u}^\varepsilon + \frac{\varepsilon^2}{2}\Delta \mathbf{u}^\varepsilon = V\mathbf{u}^\varepsilon + |\mathbf{u}^\varepsilon|^2\mathbf{u}^\varepsilon,$$

where $u^\varepsilon = u^\varepsilon(t,x) \in \mathbb{C}$ depends on the time variable $t \in \mathbb{R}$ and space variable $x \in \mathbb{R}^n$, $n \geqslant 1$. Here, ε stands for a rescaled Planck constant. We consider the asymptotic behavior of the solution u^ε as $\varepsilon \to 0$ (*semi-classical limit*). The external potential $V = V(t,x)$ is smooth and real-valued. The typical case that we have in mind is the harmonic potential. The initial data are of WKB type (phase-amplitude representation). Since the problem is nonlinear, the size of these data (in terms of ε) is crucial: if $u^\varepsilon(0,x) = \varepsilon^\kappa a_0^\varepsilon(x) e^{i\phi_0(x)/\varepsilon}$, where $a_0^\varepsilon \underset{\varepsilon\to 0}{\sim} a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots$, then we change the unknown function, to always consider initial data of order $\mathcal{O}(1)$ as $\varepsilon \to 0$. Set $u^\varepsilon = \varepsilon^{-\kappa}\mathbf{u}^\varepsilon$. We consider

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + \varepsilon^\alpha |u^\varepsilon|^2 u^\varepsilon \quad ; \quad u^\varepsilon(0,x) = a_0^\varepsilon(x) e^{i\phi_0(x)/\varepsilon}, \tag{1}$$

with $\alpha = 2\kappa \geqslant 0$. The results that we present concern two régimes: local in time WKB analysis, and global in time analysis, in the presence of point caustics.

2 WKB analysis

Detailed statements and proofs of the results presented in the present paragraph can be found in [3]. For small time at least, it is reasonable to expect the wave function u^ε to be described in terms of phase and amplitude: $u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} a(t,x)e^{i\phi(t,x)/\varepsilon}$. Plugging such an *ansatz* into (1), and ordering powers of ε so that the equation is satisfied as precisely as possible by this candidate, we find

$$\mathcal{O}\left(\varepsilon^0\right) : \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + V(t,x) = \begin{cases} 0 & \text{if } \alpha > 0, \\ -|a|^2 & \text{if } \alpha = 0. \end{cases} \quad ; \quad \phi_{|t=0} = \phi_0.$$

$$\mathcal{O}\left(\varepsilon^1\right) : \partial_t a + \nabla\phi\cdot\nabla a + \frac{1}{2}a\Delta\phi = \begin{cases} 0 & \text{if } \alpha > 1, \\ -i|a|^2 a & \text{if } \alpha = 1, \\ ?? & \text{if } \alpha < 1. \end{cases} \quad ; \quad a_{|t=0} = a_0.$$

The equation for ϕ is referred to as the *eikonal equation*, and the equation for a is a *transport equation*. Morally, for "large" α, nonlinear effects are too weak to be relevant at leading order when $\varepsilon \to 0$. Two notions of criticality then arise: for $\alpha > 1$, the transport equation is the same as in the linear case, while the nonlinearity is present for $\alpha = 1$. This is the smallest value of α for which nonlinear effects modify the leading order asymptotic of u^ε (they affect a, but not ϕ: this régime is called *weakly nonlinear*). The case $\alpha = 0$ is the "worst" possible case: the first observation is that the coupling between ϕ and a is very

* Corresponding author E-mail: Remi.Carles@math.cnrs.fr

strong. We present here this case only. In the case $V \equiv 0$, seek more precisely $u^\varepsilon \underset{\varepsilon\to 0}{\sim} (a + \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \dots)e^{i\phi/\varepsilon}$. Then the interrogation marks in the transport equation become explicit:

$$\partial_t a + \nabla\phi\cdot\nabla a + \frac{1}{2}a\Delta\phi = -2i\,\mathrm{Re}\left(\overline{a}a^{(1)}\right)a.$$

The system of equations for (ϕ, a) is not closed (no matter how many terms are included), and a strong coupling phase/main amplitude is present. These issues were resolved by E. Grenier [4] in the case $V \equiv 0$, by seeking $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ (exact equality) where $a^\varepsilon = a^\varepsilon(t,x) \in \mathbb{C}$ and $\phi^\varepsilon = \phi^\varepsilon(t,x) \in \mathbb{R}$ depend on ε. A judicious choice of the system that $(\phi^\varepsilon, a^\varepsilon)$ is required to satisfy leads to a strictly hyperbolic, symmetric, quasi-linear system (plus a skew-symmetric perturbation), for which the local in time Cauchy theory is well established in Sobolev spaces $H^s(\mathbb{R}^n)$. The asymptotic expansion for ϕ^ε and a^ε then yields the asymptotic behavior of u^ε as $\varepsilon \to 0$. We emphasize the fact that to know the leading order behavior of u^ε, it is necessary to take $a_1 = a^{(1)}_{|t=0}$ into account (see also [5]). Note that the case of higher order nonlinearities (*e.g.* quintic) is not covered by the assumptions of [4]. This issue has been resolved in [6].

The case of an external potential and of a possibly unbounded initial phase ϕ_0 can be inferred by considering the solution of the "usual" eikonal equation

$$\partial_t\phi_{\mathrm{eik}} + \frac{1}{2}|\nabla\phi_{\mathrm{eik}}|^2 + V = 0 \quad ; \quad \phi_{\mathrm{eik}}(0,x) = \phi_0(x). \tag{2}$$

If V and ϕ_0 are smooth and sub-quadratic in space (their partial derivatives of order at least two are bounded, *e.g.* harmonic potential), then (2) has a smooth solution, *locally in time* and *globally in space* (see [3]).

3 Global in time analysis in the case of the harmonic potential

In the particular case $\phi_0 = 0$ and $V = V(x) = |x|^2/2$, the solution ϕ_{eik} is explicit: $\phi(t,x) = -\frac{|x|^2}{2}\tan t$. Obviously, it becomes singular as $t \to \pi/2$. However, this does not mean that the solution to (1) becomes singular (the solution may be global in time for fixed ε). In the linear case, the solution is obviously global in time: as $t \to \pi/2$, its order of magnitude changes from $\mathcal{O}(1)$ (at time $t = 0$) to $\mathcal{O}(\varepsilon^{-n/2})$: there is a caustic at the origin at time $t = \pi/2$. In the case of the harmonic potential, this phenomenon is time periodic: a caustic is formed at the origin for $t \in \pi/2 + \pi\mathbb{Z}$ (rays of geometric optics are sinusoids, and meet at the origin periodically in time). As suggested by formal computations in [7], there should be two distinct discussions as for the relevance of the nonlinearity according to the value of the parameter α in the limit $\varepsilon \to 0$: one outside the caustic, and one near the caustic. In the present case of a cubic nonlinearity, the critical index near the point caustics is $\alpha = n$, the space dimension. The critical case $\alpha = n > 1$ follows from the more general study presented in [8]: outside the caustics, the nonlinearity is negligible as $\varepsilon \to 0$. On the other hand, near the caustics, nonlinear effects alter the behavior of u^ε at leading order. Typically, for $k \in \mathbb{N}$ and $\pi/2 + (k-1)\pi < a \leqslant b < \pi/2 + k\pi$,

$$\sup_{a\leqslant t\leqslant b}\left\| u^\varepsilon(t,x) - \frac{e^{-ink\frac{\pi}{2}}}{|\cos t|^{n/2}}\left(\mathcal{F}\circ S^k\circ\mathcal{F}^{-1}\right)a_0\left(\frac{x}{\cos t}\right)e^{-i\frac{|x|^2}{2\varepsilon}\tan t}\right\|_{L^2} \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

where $\mathcal{F}$ denotes the Fourier transform, and S^k stands for the k^{th} iterate of the scattering operator associated to the (cubic) nonlinear Schrödinger equation (without potential). As a consequence, the Cauchy problem for the propagation of Wigner measures is *ill-posed*: two wave functions with the same initial Wigner measures may propagate so that the corresponding Wigner measures are distinct at time $t = \pi$ (see [9]). In the case $1 < \alpha < n$, instabilities occur near $t = \pi/2$ (see [5]).

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