AN ASYMPTOTIC PRESERVING SCHEME BASED ON A NEW FORMULATION FOR NLS IN THE SEMICLASSICAL LIMIT∗
CHRISTOPHE BESSE†, RÉMI CARLES‡, AND FLORIAN MÉHATS§

Abstract. We consider the semiclassical limit for the nonlinear Schrödinger equation. We introduce a phase/amplitude representation given by a system similar to the hydrodynamical formulation, whose novelty consists of including some asymptotically vanishing viscosity. We prove that the system is always locally well-posed in a class of Sobolev spaces, and globally well-posed for a fixed positive Planck constant in the one-dimensional case. We propose a second order numerical scheme which is asymptotic preserving. Before singularities appear in the limiting Euler equation, we recover the quadratic physical observables as well as the wave function with mesh size and time step independent of the Planck constant. This approach is also well suited to the linear Schrödinger equation.

Key words. nonlinear Schrödinger equation, semiclassical limit, numerical simulation, asymptotic preserving, Madelung transform

AMS subject classifications. 35Q55, 65M99, 76A02, 76Y05, 81Q20, 82D50

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1. Introduction and main results.

1.1. Motivation. We consider the cubic, defocusing, nonlinear Schrödinger equation (NLS)

\[ i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = |u^\varepsilon|^2 u^\varepsilon, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \]

with WKB-type initial data

\[ u^\varepsilon(0, x) = a_0(x)e^{i\phi_0(x)/\varepsilon}, \]

the functions \(a_0\) and \(\phi_0\) being real-valued. The aim of this paper is to construct an asymptotic preserving (AP) numerical scheme for this equation in the semiclassical limit \(\varepsilon \to 0\). We seek a scheme which provides an approximation of the solution \(u^\varepsilon\) with an accuracy, for fixed numerical parameters \(\Delta t, \Delta x\), that is not degraded as the scaled Planck constant \(\varepsilon\) goes to zero. In other terms, such a scheme is consistent with (1.1) for all fixed \(\varepsilon > 0\), and when \(\varepsilon \to 0\) converges to a consistent approximation of the limit equation, which is the compressible, isentropic Euler equation (1.4). The difficulty here is that when \(\varepsilon\) is small the solution \(u^\varepsilon\) becomes highly oscillatory with respect to the time and space variables, and converges to its limit only in a weak sense.

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†Laboratoire Paul Painlevé, Université de Lille 1, Villeneuve d’Ascq 59655, France (Christophe.Besse@math.univ-lille1.fr). This author was partially supported by the French ANR fundings under the project MicroWave NT09_460489.
‡CNRS & University Montpellier 2, Montpellier 34095, France (Remi.Carles@math.cnrs.fr). This author was supported by the French ANR project SchEq (ANR-12-IS01-0005-01).
§IRMAR, Université de Rennes 1 and INRIA, IPSO Project, Rennes 35042, France (florian.mehats@univ-rennes1.fr).
In order to follow these oscillations without reducing $\Delta t$ and $\Delta x$ at the size of $\varepsilon$, which may be computationally demanding, our construction relies on a fluid reformulation of (1.1), well adapted to the semiclassical limit. Note that here the notion of an AP scheme makes sense only so long as the solution to the Euler equation remains smooth.

In kinetic theory, plasma physics, and radiative transfer, the interest in AP schemes has considerably grown in recent years since the pioneer works on the subject [19, 26, 31, 36]. Among a long list of works on this topic, we can mention [33, 28, 11, 40, 7, 20, 25, 16, 21, 39, 23]. For the Schrödinger equation, fewer works exist. In the stationary case, one can cite [3], which proposes a WKB-type transformation in order to filter out the oscillations in space. In the time-dependent case, the usual numerical methods for solving the nonlinear Schrödinger equation—finite-difference schemes [52, 24, 1, 34], time-splitting methods [47, 51, 9], relaxation schemes [8], or even symplectic methods [49]—are designed in order to guarantee the convergence of the discrete wave function $u$ when $\varepsilon > 0$ is fixed. As it is analyzed, for instance, in [45, 46] for the case of finite-differences or in [37] for Runge–Kutta schemes, these methods suffer from the oscillations in the semiclassical regime if the time and space steps are not shrunk. Note, however, that, among these methods, the status of splitting methods is particular. Indeed, in the linear case, it is shown in [5] (see also the recent review article [32]) by Wigner techniques that, if the space step $\Delta x$ has to follow the parameter $\varepsilon$ when it becomes small, the time step can be chosen independently of $\varepsilon$ (before the appearance of caustics) if we are only interested in getting a good approximation of the quadratic observables associated to $u$. Still, none of these methods is asymptotic preserving in the sense defined above.

The quantum fluid reformulation of the Schrödinger equation provides a natural framework for the construction of AP numerical methods. The Madelung transform [43] is a polar decomposition of the solution of (1.1), written as

$$u(\mathbf{t}, \mathbf{x}) = a(t, x)e^{i\phi(t, x)/\varepsilon},$$

where the amplitude $a$ and the phase $\phi$ are real-valued. Inserting this ansatz into (1.1) yields the system of quantum hydrodynamics (QHD), or Madelung equations, satisfied by $\rho$ and $\mathbf{v}$:

\begin{align}
\partial_t \rho + \text{div} (\rho \mathbf{v}) &= 0, \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \rho &= \frac{\varepsilon}{2} \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right), \\
\rho_{|t=0} &= (a_0)^2, \\
\mathbf{v}_{|t=0} &= \nabla \phi_0.
\end{align}

This system takes the form of the compressible Euler equation (with the pressure law $p(\rho) = \rho^2/2$), with an additional term testifying for the quantum character of the equation, the so-called Bohm potential $\varepsilon \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right)$. Two comments are in order. First, the term in $\varepsilon$ no longer appears as a singular perturbation and, in the limit $\varepsilon \to 0$, the quantum pressure disappears, leading formally to the compressible, isentropic Euler equation

\begin{align}
\partial_t \rho + \text{div} (\rho \mathbf{v}) &= 0, \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \rho &= 0, \\
\rho_{|t=0} &= (a_0)^2, \\
\mathbf{v}_{|t=0} &= \nabla \phi_0,
\end{align}

which is the semiclassical limit of our problem (actually valid before the appearance of shocks). Second, the fluid-like form of this system enables us to consider many numerical methods originating from computational fluid mechanics.
For the linear Schrödinger equation, the Madelung formulation is at the heart of the method of quantum trajectories (or Bohmian dynamics) (see [53]) which are based on a Lagrangian resolution of this system. In Eulerian coordinates, the work [22] also exploits this formulation to construct AP schemes for (1.3). Unfortunately, the main drawback of the QHD system is the form of the quantum potential, which becomes singular as the density $\rho$ vanishes (see [13] for a recent survey). Hence, these methods do not provide AP schemes in the presence of vacuum. Another point of view was recently developed in [15] for the nonlinear equation (1.1), taking advantage of another fluid reformulation of the equation—due to Grenier [29]—which tolerates the presence of vacuum. In the next subsection, 1.2, we present in detail this formulation relying on a complex amplitude version of the Madelung transform. However, as we will see, the drawback of this reformulation is that this system may develop singularities even for fixed $\varepsilon > 0$, whereas the solution of NLS remains smooth. Then, in the last subsection, 1.3, of this introduction, we present a modification in order to remedy this problem and preserve the regularity of the solution. The new QHD model that we propose contains a viscosity term in the equation for the velocity, but is still equivalent to the original equation (1.1).

1.2. Backgrounds on NLS and its semiclassical limit. In this section, we recall some known results on the Cauchy problem for NLS (1.1) and on its semiclassical limit as $\varepsilon \to 0$. For details on this subject, we refer to the textbooks [17] and [12].

The following result is standard in the case $\varepsilon = 1$, and can easily be deduced when $\varepsilon \in (0, 1]$, by scaling arguments.

**Proposition 1.** Let $d \leq 3$ and $u_0^\varepsilon \in H^1(\mathbb{R}^d)$.

1. There exists a unique solution $u^\varepsilon \in C(\mathbb{R}; H^1(\mathbb{R}^d)) \cap L^{8/d}_{\text{loc}}(\mathbb{R}; W^{1,4}(\mathbb{R}^d))$ to (1.1), such that $u^\varepsilon_{t=t=0} = u_0^\varepsilon$. Moreover, the following conservations hold:

- **Mass:** $\frac{d}{dt} \| u^\varepsilon(t) \|_{L^2}^2 = 0$.
- **Momentum:** $\frac{d}{dt} \int_{\mathbb{R}^d} \overline{u^\varepsilon(t,x)} \nabla u^\varepsilon(t,x) dx = 0$.
- **Energy:** $\frac{d}{dt} \left( \| \varepsilon \nabla u^\varepsilon(t) \|_{L^2}^2 + \| u^\varepsilon(t) \|_{L^4}^4 \right) = 0$.

2. If, in addition, $u_0^\varepsilon \in H^k(\mathbb{R}^d)$ for $k \in \mathbb{N}$, $k \geq 2$, then $u^\varepsilon \in C(\mathbb{R}; H^k(\mathbb{R}^d))$.

Note that, if $a^\varepsilon = a^\varepsilon e^{\phi^\varepsilon/\varepsilon}$ with $\phi \in \mathbb{R}$, the above three conservation laws become

\begin{equation}
\frac{d}{dt} \| a^\varepsilon(t) \|_{L^2}^2 = 0.
\end{equation}

\begin{equation}
\frac{d}{dt} \left( \int_{\mathbb{R}^d} \overline{a^\varepsilon(t,x)} \nabla a^\varepsilon(t,x) dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} |a^\varepsilon(t,x)|^2 \nabla \phi^\varepsilon(t,x) dx \right) = 0.
\end{equation}

\begin{equation}
\frac{d}{dt} \left( \| \varepsilon \nabla a^\varepsilon(t) + ia^\varepsilon(t) \nabla \phi^\varepsilon(t) \|_{L^2}^2 + \| a^\varepsilon(t) \|_{L^4}^4 \right) = 0.
\end{equation}

Let us now describe the limit $\varepsilon \to 0$ for (1.1). The toolbox for studying semiclassical Schrödinger equations contains a variety of methods, depending on the quantities we are interested in. If one is only interested in a description of the dynamics of quadratic observables, such as the mass, current, or energy densities, the Wigner transform is well adapted and has been applied to linear Schrödinger equations and Schrödinger–Poisson systems [27, 42, 56]; on the other hand, it is not adapted to the study of NLS [14]. Still, for a description of quadratic observables, the modulated...
energy method [10, 55, 41, 2] enables us to treat the nonlinear equation (1.1). Yet, for a pointwise description of the wavefunction \(u\), WKB techniques are more convenient. We refer to [12] for a presentation of WKB analysis of Schrödinger equations.

In the case of (1.1), also referred to as supercritical geometric optics in this context, the justification of the WKB expansion was done by Grenier [29]. Let us briefly present his idea. As we said above, the main inconvenience of the QHD system (1.3) is the singularity of the Bohm potential at the zeroes of the function \(\rho\) (i.e., at the zeroes of the wavefunction \(u\)). Looking again for solutions \(u\) under the form

\[
(u^\varepsilon)(t, x) = a^\varepsilon(t, x)e^{i\phi^\varepsilon(t, x)/\varepsilon},
\]

but now allowing \(a\) to take complex values, Grenier proposed to define \(a\) and \(\phi\) as the solutions of the system

\[
\begin{align*}
\frac{\partial}{\partial t}a^\varepsilon + \nabla \cdot \nabla a^\varepsilon + \frac{1}{2}a^\varepsilon \Delta \phi^\varepsilon &= i\varepsilon \Delta a^\varepsilon, \\
\frac{\partial}{\partial t}a^\varepsilon + \nabla \cdot \nabla a^\varepsilon + |a^\varepsilon|^2 &= 0,
\end{align*}
\]

This system, which is formally equivalent to (1.1), can also be expressed in terms of the unknowns \(a\) and \(\nu = \nabla \phi\). Differentiating with respect to \(x\) the second equation in (1.8), we find, using the fact that \(\nu\) is irrotational,

\[
\begin{align*}
\frac{\partial}{\partial t}a^\varepsilon + \nu \cdot \nabla a^\varepsilon + \frac{1}{2}a^\varepsilon \nabla \cdot \nabla a^\varepsilon &= i\varepsilon \Delta a^\varepsilon, \\
\frac{\partial}{\partial t}v^\varepsilon + \nu \cdot \nabla v^\varepsilon + \nabla |a^\varepsilon|^2 &= 0,
\end{align*}
\]

The important remark made in [29] is to notice that the above system is hyperbolic symmetric, perturbed by a skew-symmetric term. In the limit case \(\varepsilon = 0\), if \(a_0\) is real-valued and nonnegative, (1.9) is the Euler equation (1.4) written in symmetric form (see e.g., [44, 18]):

\[
\begin{align*}
\frac{\partial}{\partial t}a + v \cdot \nabla a + \frac{1}{2}a \nabla \cdot v &= 0, \\
\frac{\partial}{\partial t}v + v \cdot \nabla v + \nabla |a|^2 &= 0,
\end{align*}
\]

1.3. New model and main results. A drawback of (1.9) is that large time existence results (for fixed \(\varepsilon > 0\)) do not seem available. In particular, it is not known whether (1.9) still has a smooth solution past the time where the solution to (1.10) has ceased to be smooth (such a time necessarily exists if \(\phi_0\) and \(a_0\) are compactly supported, regardless of their size, from [44]—see also [54]). A practical consequence of this fact is that a numerical method based on (1.9), such as the one proposed in [15], may not approximate correctly the original equation (1.1) on arbitrary time intervals, for fixed \(\varepsilon > 0\).
To overcome this issue, we use the degree of freedom given by (1.7) in a manner which is slightly different from the approach introduced by Grenier. From (1.7), we have
\[
i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon - |u^\varepsilon|^2 u^\varepsilon = - \left( \partial_t \phi^\varepsilon + \frac{1}{2} |\nabla \phi^\varepsilon|^2 + |a^\varepsilon|^2 \right) a^\varepsilon e^{i\phi^\varepsilon}/\varepsilon
\]
\[
+ i\varepsilon \left( \partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon - i \frac{\varepsilon}{2} \Delta a^\varepsilon \right) e^{i\phi^\varepsilon}/\varepsilon.
\]
Introduce the viscous term \( \varepsilon^2 a^\varepsilon e^{i\phi^\varepsilon}/\varepsilon \Delta \phi^\varepsilon \), and reorder the terms so as to consider the new system
\[
\begin{align*}
\partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon &= \varepsilon a^\varepsilon + \varepsilon a^\varepsilon \Delta \phi^\varepsilon = -i \frac{\varepsilon}{2} \Delta a^\varepsilon - i \varepsilon a^\varepsilon \Delta \phi^\varepsilon, \\
\partial_t \phi^\varepsilon + \frac{1}{2} |\nabla \phi^\varepsilon|^2 + |a^\varepsilon|^2 &= \varepsilon^2 \Delta \phi^\varepsilon,
\end{align*}
\]
(a)|\( a^\varepsilon|\), \( \phi^\varepsilon|\)\).
Proceeding like above, we get the following new system:
\[
\begin{align*}
\partial_t a^\varepsilon + \varepsilon v^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \text{div } v^\varepsilon &= \varepsilon^2 \Delta a^\varepsilon - i \varepsilon a^\varepsilon \text{div } v^\varepsilon, \\
\partial_t v^\varepsilon + \varepsilon \text{div } v^\varepsilon + 2 \text{Re}(\varepsilon a^\varepsilon \nabla a^\varepsilon) &= \varepsilon^2 \Delta v^\varepsilon, \\
\end{align*}
\]
\(a^\varepsilon|\), \(v^\varepsilon|\).
We recover physical observables, respectively particle, current, and energy densities, thanks to the formula
\[
\begin{align*}
\rho^\varepsilon &= |a^\varepsilon|^2, \\
\mathbf{j}^\varepsilon &= \varepsilon \text{Im}(\overline{a^\varepsilon} \nabla a^\varepsilon) + \rho^\varepsilon v^\varepsilon, \\
e^\varepsilon &= |\varepsilon \nabla a^\varepsilon + \text{div } \mathbf{j}^\varepsilon + |a^\varepsilon|^4.
\end{align*}
\]

**Remark 2** (linear case). In the case of a linear Schrödinger equation
\[
i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = Vu^\varepsilon,
\]
we will see that the introduction of this artificial diffusion is even more striking than in the nonlinear framework; see subsection 4.1.

**Remark 3** (irreversibility). Our choice for introducing the viscous term in (1.11) makes the system irreversible (in time), while (1.1) is reversible. To solve the Schrödinger equation backward in time, it suffices to change the signs to
\[
\begin{align*}
\partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon &= i \varepsilon a^\varepsilon + i \varepsilon a^\varepsilon \Delta \phi^\varepsilon, \\
\partial_t \phi^\varepsilon + \frac{1}{2} |\nabla \phi^\varepsilon|^2 + |a^\varepsilon|^2 &= -\varepsilon^2 \Delta \phi^\varepsilon,
\end{align*}
\]
a|\( a^\varepsilon|\), \( \phi^\varepsilon|\).
Let us now present our main theoretical results on this system. Our first result concerns the local in time well-posedness of the Cauchy problem in any dimension, for fixed values of \( \varepsilon \geq 0 \).

**Theorem 4.** Let \((a_0, \phi_0) \in H^s(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d), \) with \( s > d/2 + 1 \). Then the following hold:
(i) The system (1.12) admits a unique maximal solution \((a^\varepsilon, v^\varepsilon) \in C([0, T_{\text{max}}^\varepsilon); H^s \times H^s)\).
(ii) There exists $T > 0$ independent of $\varepsilon \in [0, 1]$ such that $T_{\text{max}}^\varepsilon \geq T$. Moreover, the $L^\infty([0,T]; H^s \times H^s)$ norm of $(a^\varepsilon, v^\varepsilon)$ is bounded uniformly in $\varepsilon \in [0, 1]$.

(iii) Defining $\phi^\varepsilon \in C([0, T_{\text{max}}^\varepsilon ]; H^{s+1})$ by

\begin{equation}
\phi^\varepsilon(t, x) = \phi_0(x) - \int_0^t \left( \frac{1}{2} \nabla v^\varepsilon(s, x) \cdot \nabla (a^\varepsilon(s, x)) + \varepsilon^2 \text{div} v^\varepsilon(s, x) \right) ds,
\end{equation}

then the function

\[ u^\varepsilon = a^\varepsilon e^{i \phi^\varepsilon / \varepsilon} \in C([0, T_{\text{max}}^\varepsilon ]; H^{s}) \]

is the unique solution to (1.1) satisfying (1.2).

Remark 5. In fact, the proof of this theorem provides a criterion of global existence for the solution to (1.12). Indeed, we shall prove that the lifespan $T_{\text{max}}^\varepsilon$ is independent of $s > d/2 + 1$ and that

\begin{equation}
T_{\text{max}}^\varepsilon < \infty \iff \int_0^{T_{\text{max}}^\varepsilon } \left( \| (a^\varepsilon, v^\varepsilon)(t) \|_{W^{1, \infty}} + \| a^\varepsilon(t) \|_{L^\infty}^2 \right) dt = \infty.
\end{equation}

Our second result states the convergence of the solution to (1.12) towards the solution of the Euler equation, as $\varepsilon \to 0$.

Notation 6. For two positive numbers $\alpha^\varepsilon$ and $\beta^\varepsilon$, the notation $\alpha^\varepsilon \lesssim \beta^\varepsilon$ means that there exists $C > 0$ independent of $\varepsilon$ such that for all $\varepsilon \in (0, 1]$, $\alpha^\varepsilon \leq C \beta^\varepsilon$.

Theorem 7. Let $s > d/2 + 3$ and $(a_0, \phi_0) \in H^s \times H^{s+1}$. Let $T > 0$ such that the Euler equation (1.4) has a unique solution $(\rho, v) \in C([0, T]; H^s \times H^s)$. Then (1.10) has a unique solution $(a, v) \in C([0, T]; H^s \times H^s)$. Moreover, for $\varepsilon > 0$ sufficiently small, we have $T_{\text{max}}^\varepsilon > T$ and

\[ \| (a^\varepsilon, v^\varepsilon) - (a, v) \|_{L^\infty([0,T];H^{s-2} \times H^{s-2})} \lesssim \varepsilon. \]

Theorems 4 and 7 are not different from the corresponding results obtained by Grenier in [29] on his system (1.9). We now state a global existence result for fixed $\varepsilon > 0$ in dimension one that is not available for (1.9). This result takes advantage of the viscous term in the second equation of (1.12).

Theorem 8. Suppose $d = 1$ and let $(a_0, \phi_0) \in H^s \times H^{s+1}$, with $s \geq 2$. Then for $\varepsilon > 0$ fixed, the solution to (1.12) is global in time, in the sense that $T_{\text{max}}^\varepsilon = \infty$ in Theorem 4.

The above results show that our approach is a new way to compute the solution to (1.1) for fixed $\varepsilon > 0$ (no approximation is made in (1.11), from Theorem 4), and for all time at least for $d = 1$, a case where the solution to (1.11) is global, like the solution to (1.1). As $\varepsilon \to 0$, (1.11) yields an approximation of the Euler system, from Theorem 7, so long as the solution to (1.4) remains smooth. After the formation of singularities in (1.4), the solution to (1.11) remains smooth for fixed $\varepsilon > 0$ in the case $d = 1$, but numerics show that it becomes $\varepsilon$-oscillatory: The solution to (1.11) then no longer has a strong limit as $\varepsilon \to 0$. This aspect was already observed in [15], and is confirmed by the present experiments (see section 3). This can be understood as follows: For “large” time, describing the solution to (1.1) with only one rapid oscillation carried by a single exponential ceases to be relevant, so the amplitude/phase decomposition no longer simplifies the semiclassical analysis.

This paper is organized as follows. Section 2 is devoted to the proofs of our three theorems: the local existence result in subsection 2.1, the semiclassical limit...
in subsection 2.2, and the global existence result in subsection 2.3. Section 3 is of a different nature and concerns numerics. We describe a second order AP numerical scheme for NLS, based on the reformulation (1.12), and we give the results of numerical experiments, in dimensions 1 and 2. We show that our scheme is indeed AP before the formation of singularities in the Euler equation, which means that the time and space steps $\Delta t$ and $\Delta x$ can be taken independently of $\varepsilon$. After the formation of singularities, we show that, in order to get a good approximation of the solution, we need to take $\Delta t$ and $\Delta x$ of order $O(\varepsilon)$. At the end of this paper, in section 4, we sketch two extensions of our results: We discuss the case of the linear Schrödinger equation, and then we discuss other nonlinearities.

2. Proofs of the main theorems. This section is devoted to the proofs of our main results stated in Theorems 4, 7, and 8.

2.1. Local existence. In this subsection, we show the local existence of a smooth maximal solution to (1.12).

Proof of Theorem 4. Items (i) and (ii): Local existence and uniform estimates.

Following [29], introduce the vector-valued unknown function

$$U^\varepsilon = \begin{pmatrix} \text{Re} a_1^\varepsilon \\
\text{Im} a_1^\varepsilon \\
v_1^\varepsilon \\
\vdots \\
v_d^\varepsilon \\
v_d^\varepsilon \end{pmatrix} \in \mathbb{R}^{d+2}.$$

In terms of this unknown function, (1.12) reads as

$$\partial_t U^\varepsilon + \sum_{j=1}^d A^j(U^\varepsilon)\partial_j U^\varepsilon = \frac{\varepsilon}{2} LU^\varepsilon + \varepsilon^2 DU^\varepsilon + \varepsilon \sum_{j=1}^d B^j(U^\varepsilon)\partial_j U^\varepsilon,$$

where the $(k, \ell)_{1 \leq k, \ell \leq d+2}$ elements of the matrices $A^j(U) \in \mathcal{M}_{d+2}(\mathbb{R})$ are given by

$$
\begin{align*}
A^j_{k,k}(U) &= U_{j+2} & \text{for } k = 1, \ldots, d+2, \\
A^j_{1,j+2}(U) &= U_1/2, & A^j_{2,j+2}(U) &= U_2/2, \\
A^j_{j+2,1}(U) &= 2U_1, & A^j_{j+2,2}(U) &= 2U_2, \\
A^j_{k,\ell}(U) &= 0 & \text{otherwise.}
\end{align*}
$$

The linear operator $L$ is given by

$$L = \begin{pmatrix} 0 & -\Delta & 0 & \cdots & 0 \\
\Delta & 0 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

A first important remark is that even though $L$ is a differential operator of order two, it causes no loss of regularity in the energy estimates, since it is skew-symmetric. Also, (2.1) is hyperbolic symmetric (or symmetrizable). Indeed, let

$$S = \begin{pmatrix} I_2 & 0 \\
0 & \frac{1}{4} I_d \end{pmatrix}.$$
This matrix is symmetric and positive, and $SA^j$ is symmetric for all $k$,

$$SA^j(U) \in S_{d+2}(\mathbb{R}) \quad \forall U \in \mathbb{R}^{d+2}.$$ 

The diffusive term $D$ is given by

$$D = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & I_{d \times d} & \Delta
\end{pmatrix}. $$

Finally, the matrices $B^j$, defined analogously to $A^j$, are given by

$$B^j_{1,j+2}(U) = U_2, \quad B^j_{2,j+2}(U) = -U_1, $$

$$B^j_{k,l}(U) = 0 \text{ otherwise.} $$

When the right-hand side of (2.1) is zero, local existence of a unique solution in $H^s$ with $s > d/2 + 1$ is a consequence of standard quasi-linear analysis for hyperbolic symmetric systems; see, e.g., [50]. The important point in energy estimates consists of computing

$$\frac{d}{dt} \langle \Lambda^s(SU^\varepsilon), \Lambda^s U^\varepsilon \rangle, \quad \Lambda = (1 - \Delta)^{1/2},$$

and taking advantage of symmetry to obtain, thanks to tame estimates,

$$\frac{d}{dt} \langle \Lambda^s(SU^\varepsilon), \Lambda^s U^\varepsilon \rangle \leq C \|U^\varepsilon\|_{W^{1,\infty}} \langle \Lambda^s(SU^\varepsilon), \Lambda^s U^\varepsilon \rangle$$

for some constant $C$ depending only on $A$ and $d$.

In the case of the complete system (2.1), we first recall that as noticed in [29], the term $L$ does not alter the above energy estimate, since $SL$ is skew-symmetric, so we have

$$\frac{d}{dt} \langle \Lambda^s(SU^\varepsilon), \Lambda^s U^\varepsilon \rangle \leq C \|U^\varepsilon\|_{W^{1,\infty}} \langle \Lambda^s(SU^\varepsilon), \Lambda^s U^\varepsilon \rangle + 2\varepsilon^2 \langle SA^s DU^\varepsilon, \Lambda^s U^\varepsilon \rangle $$

$$+ 2\varepsilon \sum_{j=1}^d \langle SA^s (B^j(U^\varepsilon)\partial_j U^\varepsilon), \Lambda^s U^\varepsilon \rangle.$$ 

In terms of $(v^\varepsilon, a^\varepsilon)$, the first new term reads as

$$2\varepsilon^2 \langle SA^s DU^\varepsilon, \Lambda^s U^\varepsilon \rangle = \frac{\varepsilon^2}{2} \sum_{k=1}^d \int \Delta (\Lambda^s v_k^\varepsilon) \Lambda^s v_k^\varepsilon = -\frac{\varepsilon^2}{2} \int |\nabla \Lambda^s v^\varepsilon|^2.$$ 

For the second new term, we have

$$2\varepsilon \langle SA^s (B^j(U^\varepsilon)\partial_j U^\varepsilon), \Lambda^s U^\varepsilon \rangle = \varepsilon \int \Lambda^s (a_k^\varepsilon \partial_j v_j^\varepsilon) \Lambda^s a_1^\varepsilon = \varepsilon \int \Lambda^s (a_1^\varepsilon \partial_j v_j^\varepsilon) \Lambda^s a_2^\varepsilon.$$ 

By Kato–Ponce commutator estimates [35],

$$\|\Lambda^s (a_k^\varepsilon \partial_j v_j^\varepsilon) - a_k^\varepsilon \Lambda^s \partial_j v_j^\varepsilon\|_{L^2} \lesssim \|\nabla a_k^\varepsilon\|_{L^\infty} \|v_j^\varepsilon\|_{H^s} + \|a_k^\varepsilon\|_{H^s} \|\nabla v^\varepsilon\|_{L^\infty}. $$
Then using the Cauchy–Schwarz and Young inequalities,

\[
\int a_k^* A_k^* j v_\varepsilon \cdot \partial_j v_\varepsilon \|L^2 \leq \|A^* v\|_{L^2} \|a_k^* A_k^* j\|_{L^2} \leq \delta \varepsilon \|A^* v\|_{L^2}^2 + \frac{1}{4 \delta \varepsilon} \|a_k^* A_k^* j\|_{L^2}^2.
\]

Choosing \( \delta > 0 \) sufficiently small and independent of \( \varepsilon \), we infer

\[
\frac{d}{dt} (A^* (SU), A^* U^\varepsilon) + \varepsilon^2 \frac{2}{4} \|A^* v\|_{L^2}^2 \leq \|U_T\|_{W^{1, \infty}} \|U_T^\varepsilon\|_{H^s} + \|a^\varepsilon\|_{L^\infty}^2 \|a^\varepsilon\|_{H^s}^2.
\]

The local existence result then follows easily by adapting the standard arguments from [50]. In view of the energy estimate (2.3), we also infer (1.15). The lifespan \( T^\varepsilon_{\max} \) of the maximal solution to (1.12) must be expected to actually depend on \( \varepsilon \). For instance, if \( \phi_0, a_0 \in C_0^\infty (\mathbb{R}^d) \), then from [44] \( T^0_{\max} < \infty \), regardless of the dimension \( d \). On the other hand, we prove further (proof of Theorem 8) that if \( d = 1, T^\varepsilon_{\max} = \infty \) for all \( \varepsilon > 0 \).

**Remark 9.** It may be tempting to ask what remains of the above computations if the artificial viscosity \( \varepsilon^2 \) in (1.11) is replaced with \( \varepsilon^\theta \), for some \( \theta > 0 \). By resuming the above computations, it is easy to check that the first two points of Theorem 4 remain true provided that \( \theta \geq 2 \). This might be a threshold, in the spirit of the work [38], where the competition between vanishing dispersion and vanishing viscosity was initiated, in the case of a scalar conservation law, and a limiting behavior was emphasized.

**Item (iii):** \( u^\varepsilon \) is solution of the original NLS problem. Define \( \phi^\varepsilon \) by (1.14). Let us prove that \((a^\varepsilon, \phi^\varepsilon) \in C([0, T^\varepsilon_{\max}); H^s \times H^{s+1}) \) solves the system (1.11). Since \((a^\varepsilon, \phi^\varepsilon) \in C([0, T^\varepsilon_{\max}); H^s \times H^s) \), we readily have \( \phi^\varepsilon \in C([0, T^\varepsilon_{\max}); H^{s+1}) \). Moreover, \( \phi^\varepsilon \) is irrotational: Indeed, \( \text{curl} v^\varepsilon \) satisfies a homogeneous equation (see, e.g., [4, p. 291]), and it is zero at time \( t = 0 \), so \( \text{curl} v^\varepsilon \equiv 0 \). Set \( \Omega^\varepsilon = Dv^\varepsilon - \nabla v^\varepsilon \), where \( Dv^\varepsilon \) stands for the Jacobian matrix of \( v^\varepsilon \), and \( \nabla v^\varepsilon \) stands for its transposed matrix. It solves

\[
\partial_t \Omega^\varepsilon + v^\varepsilon \cdot \nabla \Omega^\varepsilon + \Omega^\varepsilon \cdot Dv^\varepsilon + \nabla v^\varepsilon \cdot \Omega^\varepsilon = \varepsilon^2 \Delta \Omega^\varepsilon.
\]

As a consequence, \( \nabla |v^\varepsilon|^2 = 2v^\varepsilon \cdot \nabla v^\varepsilon \). We then deduce from (1.14) and from the second equation of (1.12)

\[
\partial_t (\nabla \phi^\varepsilon - v^\varepsilon) = \nabla \partial_t \phi^\varepsilon - \partial_t v^\varepsilon = 0,
\]

so we infer from the initial condition \( v^\varepsilon = \nabla \phi^\varepsilon \) that \( v^\varepsilon = \nabla \phi^\varepsilon \). Therefore, \( \nabla \phi^\varepsilon \in C([0, T^\varepsilon_{\max}); H^s) \), and hence

\[
\phi^\varepsilon \in C([0, T^\varepsilon_{\max}); H^{s+1}).
\]

Replacing \( v^\varepsilon \) with \( \nabla \phi^\varepsilon \) in (1.14) shows that \( \phi^\varepsilon \) solves the first equation in (1.11), and the second equation in (1.11) is obviously satisfied.

Now, define \( u^\varepsilon \) by (1.7). The property \( u^\varepsilon \in C([0, T^\varepsilon]; H^s) \) is straightforward, since \( H^s (\mathbb{R}^d) \) is an algebra. It is clear that \( u^\varepsilon \) solves (1.1) and (1.2). Since

\[
C([0, T^\varepsilon]; H^s (\mathbb{R}^d)) \subset C([0, T^\varepsilon]; H^1 (\mathbb{R}^d)) \cap L^{8/d}(0, T^\varepsilon); W^{1, 4} (\mathbb{R}^d)),
\]

uniqueness stems from Proposition 1.  \( \Box \)
2.2. Convergence towards the Euler equation. In this subsection, we study the semiclassical limit of (1.12).

Proof of Theorem 7. The proof proceeds as for [29, Theorem 1.2]. First, Theorem 4 yields a solution \((a, v) \in C([0, T^0_{\text{max}}]; H^s \times H^s)\). Necessarily, \(T^0_{\text{max}} > T\) (where \(T\) is an existence time for the Euler equation), for if we had \(T^0_{\text{max}} \leq T\), then by uniqueness for (1.4), \((|a|^2, v) = (\rho, v) \in L^\infty([0, T]; H^s \times H^s)\). The equation for \(a\) in (1.10) then shows that \(a \in L^\infty([0, T]; H^{s-1})\), and hence \((a, v) \in L^\infty([0, T]; H^{s-1} \times H^s)\), which is impossible if \(T^0_{\text{max}} \leq T\): This yields the first part of the theorem.

To prove the error estimate, introduce the vector-valued unknown \(U\), associated to \((a, v)\). We know that \(U \in C([0, T]; H^s)\). Consider the error \(W^\varepsilon = U^\varepsilon - U\). It solves

\[
\partial_t W^\varepsilon + \sum_{j=1}^d (A_j^\varepsilon U^\varepsilon) \partial_j U^\varepsilon - A_j^\varepsilon \partial_j U = \frac{\varepsilon}{2} L U^\varepsilon + \varepsilon^2 D U^\varepsilon + \varepsilon \sum_{j=1}^d B_j^\varepsilon U^\varepsilon.
\]

Rewrite this equation as

\[
\partial_t W^\varepsilon + \sum_{j=1}^d A_j^\varepsilon \partial_j W^\varepsilon = - \sum_{j=1}^d (A_j^\varepsilon U^\varepsilon - A_j^\varepsilon U) \partial_j U + \frac{\varepsilon}{2} L W^\varepsilon + \varepsilon^2 D U^\varepsilon + \varepsilon \sum_{j=1}^d B_j^\varepsilon U^\varepsilon.
\]

The operator on the left-hand side is the same operator as in (2.1). The term \(L W^\varepsilon\) is not present in the energy estimates, since it is skew-symmetric. The term \(\varepsilon L U\) is considered as a source term: It is of order \(\varepsilon\), uniformly in \(L^\infty([0, T]; H^{s-2})\). Next, the first term on the right-hand side is a semilinear perturbation:

\[
\left\| (A_j^\varepsilon U^\varepsilon - A_j^\varepsilon U) \partial_j U \right\|_{H^{s-2}} \leq \left\| (A_j^\varepsilon U^\varepsilon - A_j^\varepsilon U) \right\|_{H^{s-2}} \left\| U \right\|_{H^{s-1}} \leq C \left\| W^\varepsilon \right\|_{L^\infty} \left\| U \right\|_{H^{s-1}} \left\| W^\varepsilon \right\|_{H^{s-2}},
\]

where we have used tame estimates. Finally, we know that \(W^\varepsilon\) is bounded in \(L^\infty([0, T] \times \mathbb{R}^d)\), as the difference of two bounded terms. The terms involving \(D\) and \(B\) are treated in a similar way, by adapting the estimates used in the proof of Theorem 4. We end up with

\[
\frac{d}{dt} \left( \langle S \Lambda^{s-2} W^\varepsilon, \Lambda^{s-2} W^\varepsilon \rangle + \frac{\varepsilon^2}{4} \left\| \nabla \Lambda^{s-2} W^\varepsilon \right\|_{L^2} \right) \leq \varepsilon \left\| W^\varepsilon \right\|_{H^{s-2}} + \left\| W^\varepsilon \right\|_{H^{s-2}}^2 \lesssim \varepsilon^2 + \left\langle S \Lambda^{s-2} W^\varepsilon, \Lambda^{s-2} W^\varepsilon \right\rangle,
\]

and the result follows from the Gronwall lemma.

2.3. Global existence result in one dimension. In this section, we consider (1.12) with \(\varepsilon > 0\) fixed, in dimension \(d = 1\). To prove a global existence result, we shall need finer estimates than (2.3), in particular for low order derivatives.

Proof of Theorem 8. For \(s = 0\) and \(s = 1\), (1.12) can be refined, and this will be useful to prove a global existence result.

\(L^2\) estimates. Multiplying the second equation in (1.12) by \(a^\varepsilon\), integrating in space, and considering the real part yields

\[
\frac{d}{dt} \left\| a^\varepsilon(t) \right\|_{L^2}^2 = 0.
\]
We have simply recovered the conservation of mass (1.5).

Multiplying now the first equation in (1.12) by \( v^\varepsilon \) and integrating in space, we take advantage of the one dimension:

\[
\frac{1}{2} \frac{d}{dt} \|v^\varepsilon(t)\|_{L^2}^2 + \varepsilon^2 \|\partial_x v^\varepsilon(t)\|_{L^2}^2 \lesssim \int |a^\varepsilon(t, x)|^2 |\partial_x v^\varepsilon(t, x)| dx. \tag{2.4}
\]

\[ \dot{H}^1 \text{ estimates.} \] Differentiating (1.12) in space, then multiplying the first equation by \( \partial_x v^\varepsilon \) and the second by \( \partial_x a^\varepsilon \), and integrating, we find (thanks to the symmetrizer)

\[ \frac{d}{dt} \left( \|\partial_x v^\varepsilon(t)\|_{L^2}^2 + 4 \|\partial_x a^\varepsilon(t)\|_{L^2}^2 \right) + \varepsilon^2 \|\partial_x^2 v^\varepsilon(t)\|_{L^2}^2 \lesssim \left( \|\partial_x v^\varepsilon(t)\|_{L^\infty} + \|a^\varepsilon(t)\|_{L^\infty} \right) \left( \|v^\varepsilon(t)\|_{\dot{H}^1} + \|a^\varepsilon(t)\|_{\dot{H}^1} \right). \tag{2.5} \]

Now, we have the tools to prove Theorem 8. In view of Remark 5, it suffices to prove that for any fixed \( \varepsilon > 0 \),

\[ v^\varepsilon, \partial_x v^\varepsilon, a^\varepsilon, \partial_x a^\varepsilon, |a^\varepsilon|^2 \in L^1_{loc} \left( [0, \infty); L^\infty(\mathbb{R}^d) \right). \]

Since \( \varepsilon > 0 \) is fixed, we shall omit it in the notations, and fix it equal to 1 in (1.12).

Using item (iii) of Theorem 4, and in view of (1.6), we have the a priori estimate

\[ \|a(t)\|_{L^4} \lesssim 1 \quad \forall t \geq 0. \]

The refined \( L^2 \) estimate (2.4) for \( v \) and the Cauchy–Schwarz inequality yield

\[ \frac{1}{2} \frac{d}{dt} \|v(t)\|_{L^2}^2 + \|\partial_x v(t)\|_{L^2}^2 \lesssim \|a(t)\|_{L^4}^2 \|\partial_x v(t)\|_{L^2} \lesssim \|\partial_x v(t)\|_{L^2}. \]

Using Young’s inequality

\[ \|\partial_x v(t)\|_{L^2} \lesssim \delta \|\partial_x v(t)\|_{L^2}^2 + \frac{1}{4\delta}, \]

we infer, for \( \delta > 0 \) sufficiently small,

\[ \frac{d}{dt} \|v(t)\|_{L^2}^2 + \|\partial_x v(t)\|_{L^2}^2 \lesssim 1. \]

Therefore,

\[ \|v(t)\|_{L^2}^2 + \int_0^t \|\partial_x v(\tau)\|_{L^2}^2 d\tau \lesssim 1 + t. \]

The Gagliardo–Nirenberg inequality yields

\[ \int_0^t \|v(\tau)\|_{L^\infty} d\tau \lesssim \int_0^t \|v(\tau)\|_{L^2}^{1/2} \|\partial_x v(\tau)\|_{L^2}^{1/2} d\tau \lesssim (1 + t)^{1/4} \int_0^t \|\partial_x v(\tau)\|_{L^2} d\tau \]

\[ \lesssim (1 + t)^{1/4} \int_0^t (1 + \|\partial_x v(\tau)\|_{L^2}^2) d\tau \lesssim (1 + t)^{5/4}, \]

where we have used the Young inequality. In view of (1.6), we infer

\[ \|\partial_x a(t)\|_{L^2} \lesssim 1 + \|a(t)v(t)\|_{L^2} \lesssim 1 + \|v(t)\|_{L^\infty}, \]
We infer
\[
\int_0^t \left( \|a(\tau)\|_{L^\infty} + \|a(\tau)\|_{L^2}^2 \right) d\tau \lesssim (1 + t)^{5/4}.
\]

It only remains to prove that \( \partial_x v, \partial_x a \in L^1_{\text{loc}}([0, \infty); L^\infty(\mathbb{R})) \). The \( \dot{H}^1 \) estimate (2.5) yields
\[
\int_0^t \|\partial_x^2 v(\tau)\|_{L^2}^2 d\tau \lesssim 1 + \int_0^t \left( \|\partial_x v(\tau)\|_{L^\infty} + 1 + \|a(\tau)\|_{L^\infty}^2 \right) \left( \|a(\tau)\|_{H^1}^2 + \|v(\tau)\|_{H^1}^2 \right) d\tau \\
\lesssim 1 + \int_0^t \left( \|\partial_x v(\tau)\|_{L^\infty} + 1 + \|a(\tau)\|_{L^\infty}^2 \right) \left( 1 + \|v(\tau)\|_{L^\infty}^2 + \|v(\tau)\|_{H^1}^2 \right) d\tau \\
\lesssim 1 + \int_0^t \left( \|\partial_x v(\tau)\|_{L^\infty} + 1 + \|a(\tau)\|_{L^\infty}^2 \right) \left( 1 + \|v(\tau)\|_{H^1}^2 \right) d\tau,
\]

from Sobolev embedding. Using an integration by parts and the Young inequality, we find
\[
\|\partial_x v(\tau)\|_{L^2}^2 \leq \|v(\tau)\|_{L^2} \|\partial_x^2 v(\tau)\|_{L^2} \leq \frac{\delta}{\|\partial_x v(\tau)\|_{L^\infty} + 1 + \|a(\tau)\|_{L^\infty}^2} \|\partial_x^2 v(\tau)\|_{L^2}^2 \\
+ \frac{\|\partial_x v(\tau)\|_{L^\infty} + 1 + \|a(\tau)\|_{L^\infty}^2}{4\delta} \|v(\tau)\|_{L^2}^2.
\]

Taking \( \delta > 0 \) sufficiently small, we infer
\[
\int_0^t \|\partial_x^2 v(\tau)\|_{L^2}^2 d\tau \lesssim 1 + (1 + t) \int_0^t \left( \|\partial_x v(\tau)\|_{L^\infty} + 1 + \|a(\tau)\|_{L^\infty}^2 \right) d\tau \\
+ (1 + t) \int_0^t \left( \|\partial_x v(\tau)\|_{L^\infty} + 1 + \|a(\tau)\|_{L^\infty}^2 \right)^2 d\tau \\
\lesssim 1 + t^2 + (1 + t) \int_0^t \|\partial_x v(\tau)\|_{L^\infty}^2 d\tau + (1 + t) \int_0^t \|a(\tau)\|_{L^\infty}^4 d\tau.
\]

We also have
\[
\int_0^t \|a(\tau)\|_{L^\infty} d\tau \lesssim t + \int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau \lesssim t + \int_0^t \|v(\tau)\|_{L^2} \|\partial_x v(\tau)\|_{L^2} d\tau \\
\lesssim t + \sqrt{1 + t} \int_0^t \|\partial_x v(\tau)\|_{L^2} d\tau \lesssim (1 + t)^{3/2},
\]

and so
\[
\int_0^t \|\partial_x^2 v(\tau)\|_{L^2}^2 d\tau \lesssim (1 + t)^{5/2} + (1 + t) \int_0^t \|\partial_x v(\tau)\|_{L^\infty}^2 d\tau.
\]
Now the Gagliardo–Nirenberg and Cauchy–Schwarz inequalities yield
\[ \int_0^t \| \partial_x v(\tau) \|_{L^\infty}^2 d\tau \lesssim \int_0^t \| \partial_x v(\tau) \|_{L^2} \| \partial_x^2 v(\tau) \|_{L^2} d\tau. \]
\[ \lesssim \left( \int_0^t \| \partial_x v(\tau) \|_{L^2}^2 d\tau \right)^{1/2} \left( \int_0^t \| \partial_x^2 v(\tau) \|_{L^2}^2 d\tau \right)^{1/2}. \]
\[ \lesssim \sqrt{1+t} \left( (1+t)^{5/2} + (1+t) \int_0^t \| \partial_x v(\tau) \|_{L^\infty}^2 d\tau \right)^{1/2}. \]
\[ \lesssim (1+t)^{7/4} + (1+t) \left( \int_0^t \| \partial_x v(\tau) \|_{L^\infty}^2 d\tau \right)^{1/2}. \]

The Young inequality implies
\[ \int_0^t \| \partial_x v(\tau) \|_{L^\infty}^2 d\tau \lesssim (1+t)^2, \]
and hence \( \partial_x v \in L^2_\text{loc}([0, \infty); L^\infty) \subset L^1_\text{loc}([0, \infty); L^\infty). \)
Since \( \partial_x a \in L^1_\text{loc}([0, \infty); L^2) \), Gagliardo–Nirenberg shows that it suffices to prove that \( \partial_x^2 a \in L^1_\text{loc}([0, \infty); L^2) \) to conclude that \( \partial_x a \in L^1_\text{loc}([0, \infty); L^\infty) \). For that, we use again item (iii) of Theorem 4, from which we know that \( ae^{i\phi} \in L^1_\text{loc}([0, \infty); H^2) \).
Differentiating \( ae^{i\phi} \) twice, we have, since \( \partial_x \phi = v \),
\[ \| \partial_x^2 a(t) \|_{L^2} \lesssim \| ae^{i\phi} \|_{H^2} + \| v \partial_x a \|_{L^2} + \| a \partial_x v \|_{L^2} + \| v^2 a \|_{L^2} \]
\[ \lesssim \| ae^{i\phi} \|_{H^2} + \| v \|_{L^\infty} \| \partial_x a \|_{L^2} + \| a \|_{L^\infty} \| \partial_x v \|_{L^2} + \| v \|_{L^\infty}^2. \]
From the above estimate, we infer \( \partial_x^2 a \in L^1_\text{loc}([0, \infty); L^2) \), which completes the proof of the theorem. \( \square \)

Remark 10 (higher dimension). Even though we have obviously taken advantage of the one-dimensional setting to use the embedding \( H^1 \subset L^\infty \), the true reason why Theorem 8 is limited to \( d = 1 \) lies elsewhere. The refined \( L^2 \) estimate (2.4) becomes, if \( d \geq 2 \),
\[ \frac{1}{2} \frac{d}{dt} \| v^\varepsilon(t) \|^2_{L^2} + \varepsilon^2 \| \nabla v^\varepsilon(t) \|^2_{L^2} \lesssim \int \left( |a^\varepsilon(t,x)|^2 + |v^\varepsilon(t,x)|^2 \right) |\nabla v^\varepsilon(t,x)| dx. \]
Note the appearance of the new term \( \int |v^\varepsilon(t,x)|^2 |\nabla v^\varepsilon(t,x)| dx \): It corresponds to the fact that \( \int v^\varepsilon \cdot (v^\varepsilon \cdot \nabla v^\varepsilon) \) is zero if \( d = 1 \), but not if \( d \geq 2 \) (in general). This aspect seems to be the only real limitation to extend Theorem 8 to \( d \geq 2 \).

3. Numerics. In this section, we define a second order numerical scheme for (1.12). Since \( \varepsilon \) is no longer a singular perturbation parameter in this reformulation of NLS, this scheme is naturally asymptotic preserving for our original problem in the semiclassical limit \( \varepsilon \to 0 \), as long as the solution of the Euler equation remains smooth. Since we solve our problem in dimension 1 on a bounded interval and in dimension 2 on a rectangle, we add periodic boundary conditions to (1.1) or to (1.12), both formulations remaining equivalent.

3.1. The AP numerical scheme. The nonlinear system to be solved reads as
\[
\begin{cases}
\partial_t a + \text{div}(av) + \left( i\varepsilon - \frac{1}{2} \right) a \text{div} v = i\varepsilon \frac{\varepsilon}{2} \Delta a; & a_{t=0} = a_0, \\
\partial_t v + \nabla \left( \frac{|v|^2}{2} + |a|^2 \right) = \varepsilon^2 \Delta v; & v_{t=0} = v_0.
\end{cases}
\]
The semidiscretization in time is realized through a Strang splitting scheme. Let us denote by $\Delta t_n = t_n - t_{n-1}$ the variable time step, such that $t_n = \sum_{k=1}^n \Delta t_k$. On each time step $[t^n, t^{n+1}]$, we split (3.1) into two subsystems and apply the second order Strang splitting algorithm.

Step 1 for $t \in [t_n, t_n + \frac{\Delta t_{n+1}}{2}]$:
\[
\begin{align*}
\partial_t a_1 &= i\frac{\varepsilon}{2}\Delta a_1; \quad a_{1|t=t_n} = a_{1|t=t_n}, \\
\partial_t v_1 &= \varepsilon^2 \Delta v_1; \quad v_{1|t=t_n} = v_{1|t=t_n}.
\end{align*}
\]

Step 2 for $t \in [t_n, t_n + \Delta t_{n+1}]$:
\[
\begin{align*}
\partial_t a_2 + \text{div}(a_2 v_2) + \left(i\varepsilon - \frac{1}{2}\right)a_2 \text{div} v_2 &= 0; \quad a_{2|t=t_n} = a_{2|t=t_n + \frac{\Delta t_{n+1}}{2}}, \\
\partial_t v_2 + \nabla_x \left(\frac{|v_2|^2}{2} + |a_2|^2\right) &= 0; \quad v_{2|t=t_n} = v_{2|t=t_n + \frac{\Delta t_{n+1}}{2}}.
\end{align*}
\]

Step 3 for $t \in [t_n, t_n + \frac{\Delta t_{n+1}}{2}]$:
\[
\begin{align*}
\partial_t a_3 &= i\frac{\varepsilon}{2}\Delta a_3; \quad a_{3|t=t_n} = a_{2|t=t_n+1}, \\
\partial_t v_3 &= \varepsilon^2 \Delta v_3; \quad v_{3|t=t_n} = v_{2|t=t_n+1}.
\end{align*}
\]

Then, $(a_3, v_3)_{|t=t_n + \frac{\Delta t_{n+1}}{2}}$ is an approximation of $(a, v)_{|t=t_{n+1}}$, solution to (3.1).

This splitting scheme enables us to decouple the fluid part (Step 2) from the parabolic and Schrödinger parts (Steps 1 and 3), which allows a standard treatment of each step. Note that for $\varepsilon = 0$, we recover exactly the Euler equations (1.10). We denote by $(a^{n,\varepsilon}, v^{n,\varepsilon})$ the numerical solution at time $t_n$. In the spirit of [30], the following result can be proven.

PROPOSITION 11. Under the assumptions of Theorem 4, there exist $\varepsilon_0 > 0$ and $C, c_0$ independent of $\varepsilon \in [0, \varepsilon_0]$ such that for all $n \in N$ such that $t_n \in [0, T]$, where $T$ is as in Theorem 4(ii), for all $\Delta t_n \in (0, c_0)$,
\[
\begin{align*}
\|a^{n,\varepsilon}\|^2 - \rho^\varepsilon(t_n)\|_{L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)} &\leq C\left(\max_n\Delta t_n\right)^2, \\
\varepsilon \text{Im}(\overline{a^{n,\varepsilon}}\nabla a^{n,\varepsilon}) + |a^{n,\varepsilon}|^2 v^{n,\varepsilon} - j^\varepsilon(t_n)\|_{L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)} &\leq C\left(\max_n\Delta t_n\right)^2.
\end{align*}
\]

We assume, without loss of generality, that we are in a periodic framework for the space variable. This allows us to solve Steps 1 and 3 in a spectral way by using the fast Fourier transform, whereas for Step 2, we use a Lax–Wendroff scheme (with directional splitting in dimension 2). The extension to Dirichlet or Neumann boundary conditions would require one to consider sine or cosine transforms in place of FFT. The total scheme is consistent with (3.1), and second order in time and space. Since the Lax–Wendroff scheme is explicit, a Courant–Friedrich–Lewy condition is necessary for stability:
\[
\Delta t_n \leq \text{CFL} \frac{\Delta x}{\max_j \left(|u^*_j| + |a^*_j|\right)},
\]
where $v^*_{i_j}$ denotes the approximation to $v(t_n, x_j)$, $x_j$ being a node of the mesh discretization of the computational domain $\Omega$, and $\Delta x$ is the space discretization parameter. In all our numerical experiments, we take $\text{CFL} = 0.8$. 
3.2. Numerical experiments in dimension 1. In order to analyze the behavior of the numerical solutions (3.1), we proceed like in [6], and compare the behavior of numerical physical observables (1.13) computed by our AP numerical scheme to a reference solution. Since exact solutions to (1.1) with initial data (1.2) are not available, we numerically generated one thanks to the usual splitting method described in [6], applied to (1.1)–(1.2) with a meshing strategy that ensures that the time step \( \Delta t \lesssim \varepsilon \) and the space step \( \Delta x \lesssim \varepsilon \). We consider the initial data

\[
\begin{align*}
a_0(x) &= e^{-25(x-0.5)^2}, \\
v_0(x) &= -\frac{1}{2} \partial_x \ln (e^5(x-0.5) + e^{-5(x-0.5)}),
\end{align*}
\]

with \( x \in [-1/2, 3/2] \).

Remark 12 (discontinuity and periodicity). The above example is borrowed from [6]. The velocity \( v_0 \) given by (3.2) is not continuous at the boundary, when considered in a periodic setting. First, we remark that this discontinuity will not be seen when looking at the current density or the wave function itself, since the amplitude decays exponentially to zero before the boundary. A more subtle argument is the following. In the neighborhood of the boundary, since \( a \) is very small at least for small times, we are actually solving the viscous Burgers equation on \( v \), with a possibly very small viscosity \( \varepsilon^2 \). However, it is noteworthy that the discontinuity on the initial data (3.2) is increasing (since \( v_0(3/2-) = - \tanh(5) \approx -1 \), and \( v_0(3/2+) = v_0(-1/2+) = \tanh(5) \approx 1 \), where the first equality is due to the periodicity), so the solution of the Riemann problem associated to the nonviscous Burgers equation is a rarefaction wave, and this continuous function (for \( t > 0 \)) is correctly handled by the Lax–Wendroff numerical scheme.

In all our simulations, the smallest value of \( \varepsilon \) is \( 10^{-4} \). We then discretize the \( x \)-interval with \( J = 2^{15} \) subintervals and the time step is \( \Delta t = \varepsilon / 100 \). We refer to the reference solution in a generic way by the notation \( u_{ref}^\varepsilon \). With the considered initial datum, new oscillations appear in the reference solution past a time \( T^* \approx 0.10 \), meaning that singularities were created in the limit Euler system (1.10). In Figures 1 and 2, we plot particle and current densities \( \rho_{ref}^\varepsilon \) and \( j_{ref}^\varepsilon \) for \( \varepsilon = 10^{-4} \), respectively, at times \( t = 0.05 \) and \( t = 0.13 \), before and past the formation of shocks. We present in Figure 3 a zoom close to singularities which shows oscillations of physical observables.

We now emphasize the interest of the viscosity term in our new system (1.12), compared to the system (1.9) introduced by Grenier. Numerical evidence allows us to conjecture that, for fixed \( \varepsilon > 0 \), the dispersive terms in (1.9) are not sufficient to prevent the formation of singularities. In Figures 4 and 5, we plot the real and imaginary values of the function \( a^\varepsilon \), and the function \( v^\varepsilon \), computed at time \( t = 0.3105 \) for \( \varepsilon = 1 \), with fine time and space steps, for our model (1.12) (in blue in the electronic version) and for the system of Grenier (in red in the electronic version). It appears clearly that, while the solution of our system remains smooth, the solution of (1.9) has formed a discontinuity on the function \( v \), associated to a discontinuity on the derivative of \( \text{Im}(a^\varepsilon) \). In Figure 6, we plot the particle and current densities, at the same time, which enables us to check that both models remain reformulations of the same NLS equation.

In order to show the AP property of our scheme, we compute relative \( l^1 \) errors at time \( t_n \) using the formula

\[
\text{err}_{\rho^\varepsilon}(t_n) = \| \rho_{ref}^\varepsilon - \rho_n^{\varepsilon} \|_1 / \| \rho_{ref}^\varepsilon \|_1, \quad \rho_n^{\varepsilon} = |a_j^{n,\varepsilon}|^2, \\
\text{err}_{j^\varepsilon}(t_n) = \| j_{ref}^\varepsilon - j_n^{\varepsilon} \|_1 / \| j_{ref}^\varepsilon \|_1, \quad j_n^{\varepsilon} = \varepsilon \text{Im}(a_j^{n,\varepsilon} \nabla a_j^{n,\varepsilon}) + r_j^{n,\varepsilon} j_j^{n,\varepsilon},
\]

where \( \text{err}_{\rho^\varepsilon}(t_n) \) and \( \text{err}_{j^\varepsilon}(t_n) \) denote the relative errors in the particle and current densities, respectively.
FIG. 1. Particle and current densities at time $T^* = 0.05$ and for $\varepsilon = 10^{-4}$.

FIG. 2. Particle and current densities at time $T^* = 0.13$ and for $\varepsilon = 10^{-4}$.

FIG. 3. Zoom around singularity of density and current at time $T^* = 0.13$ and for $\varepsilon = 10^{-4}$.

where the $\ell^1$ norm is

$$\|u\|_1 = \Delta x \sum_{k=1}^{J-1} |u_j|.$$
Fig. 4. Function \( a^\varepsilon \) at time \( t = 0.3105 \) and for \( \varepsilon = 1 \), associated to our system (1.12) (red, in the electronic version, curves) and to the system of Grenier (1.9) (blue, in the electronic version, curves).

Fig. 5. Velocity \( v^\varepsilon \) at time \( t = 0.3105 \) and for \( \varepsilon = 1 \), associated to our system (1.12) (red, in the electronic version, curve) and to the system of Grenier (1.9) (blue, in the electronic version, curve).

Fig. 6. Particle and current densities at time \( t = 0.3105 \) and for \( \varepsilon = 1 \), associated to our system (1.12) (red, in the electronic version, curves) and to the system of Grenier (1.9) (blue, in the electronic version, curves).
We evaluate the error function for different values of $J = 2^M$, where $M$ is an integer here chosen in $[4, 13]$, and various scaled Planck parameter $\varepsilon$ in $[10^{-4}, 10^0]$. We plot in Figures 7, 8, 9, and 10 the relative error on physical observables $\rho^{\varepsilon}$ and $j^{\varepsilon}$ computed with our AP schemes at times $t = 0.05$ and $t = 0.13$, respectively, before and after the formation of singularities. We clearly see that the error is proportional to $(\Delta x)^2$, independently of $\varepsilon$ before formation of singularities. Indeed, the mean slope of the error curves with respect to $\Delta x$ is close to the value 2 (subfigure (a)). The independence with respect to $\varepsilon$ appears in subfigure (b) where error curves are flat. After the formation of shocks, the behavior of our scheme is very different and the mesh parameters have to be reduced as $\varepsilon \to 0$ to get good accuracy.

For a comparison, we present the same study for the classical time splitting scheme applied to (1.1)–(1.2) in Figures 11, 12, 13, and 14. Thanks to the spectral accuracy of Fourier methods, the error levels are smaller than for our AP scheme which is only second order with respect to time and space variables. However, contrary to our AP scheme, the error curves clearly depend on $\varepsilon$, the error being of order $O(1)$ when $\varepsilon$ is smaller than $2 \times 10^{-3}$. In order to have good accuracy, it is required to take $\Delta x \lesssim \varepsilon$.

We present on Figure 15 the evolution of $\text{err}_{\rho^\varepsilon}$ for our AP scheme with respect to

Fig. 7. $\text{err}_{\rho^\varepsilon}$ ($t = 0.05$) for AP scheme.

Fig. 8. $\text{err}_{j^\varepsilon}$ ($t = 0.05$) for AP scheme.
Fig. 9. errρ (t = 0.13) for AP scheme.

Fig. 10. errρ (t = 0.13) for AP scheme.

Fig. 11. errρ (t = 0.05) for time splitting scheme.
Fig. 12. \( \text{err}_J(t = 0.05) \) for time splitting scheme.

Fig. 13. \( \text{err}_\rho(t = 0.13) \) for time splitting scheme.

Fig. 14. \( \text{err}_J(t = 0.13) \) for time splitting scheme.
the discretization parameter $\Delta x$ for various final times and various values of $\varepsilon$. The formation time of singularities is close to $t^* = 0.12$. We clearly see that before $t^*$, $\Delta x$ can be chosen independently of $\varepsilon$ to get a fixed accuracy. This is not, however, the case past $t^*$. It suggests that we have to take $\Delta x = o(\varepsilon)$ to get good accuracy.

To make our presentation complete, we present the reconstruction of $u^\varepsilon$ thanks to (1.14) with $\varepsilon = 0.005$. We, therefore, compute the phase $\phi^\varepsilon$ with amplitude $a^\varepsilon$ and velocity $v^\varepsilon$. Since we have access to values of these quantities at every discrete time $t_n$, we approximate the time dependent integral using a simple rectangular quadrature. We see that we can recover a very good approximation of the wave function $u^\varepsilon$ with very few points before the formation of singularities (see Figure 16). Obviously, one needs more points to have a good reconstruction of the wave function past the singularities (see Figures 17 and 18).

**3.3. Numerical experiments in dimension 2.** The previous subsection was devoted to one-dimensional simulations. We present here the equivalent error analysis for the two-dimensional case. The computational domain is now the square $[-0.5, 1.5]^2$ discretized with $J$ points in each direction. We still need a reference solution generated with a Strang splitting scheme applied to (1.1)–(1.2). We take $J_{ref} = 2^{13} = 8192$, so $\Delta x \sim 2.5 \times 10^{-4}$ and $\Delta t = \varepsilon/100$. In a first test, the initial datum is related to the
Fig. 16. \( \text{Re}(u^\varepsilon) \) at \( T = 0.05 \) for \( \varepsilon = 0.005 \). In blue (in the electronic version), the reference solution and in red (in the electronic version), the solution computed with our scheme.

Fig. 17. \( \text{Re}(u^\varepsilon) \) at \( T = 0.13 \) for \( \varepsilon = 0.005 \). In blue (in the electronic version), the reference solution and in red (in the electronic version), the solution computed with our scheme.

Fig. 18. \( \text{Re}(u^\varepsilon) \) at \( T = 0.13 \) for \( \varepsilon = 0.005 \). In blue (in the electronic version), the reference solution and in red (in the electronic version), the solution computed with our scheme.
one chosen for the one-dimensional case:

\[ a_0(x, y) = e^{-25 r^2}, \]
\[ v_0(x) = -\frac{1}{5} \nabla \ln (e^{5r} + e^{-5r}), \]

where \( r = \sqrt{(x - 0.5)^2 + (y - 0.5)^2} \). We evaluate the error function for different values of \( J = 2^M \), where \( M \) is chosen in \([4, 11]\), and various scaled Planck parameter \( \varepsilon \) in \([5 \times 10^{-4}, 10^0]\). The formation time of singularities is, as in the one-dimensional situation, located around 0.10. The reference particle density and the reference modulus of the current densities for \( \varepsilon = 5 \times 10^{-4} \) are represented on Figures 19 and 20 before and after formation of singularities. We clearly see that a tiny front appears after shocks in both particle and current densities. The wave function presents strong oscillations with new ones created past the shock creation (see Figure 21).

The error analysis results are equivalent to the ones obtained for one-dimensional simulation. We recover an independence of the error with respect to \( \varepsilon \) before the
formation of singularities (see Figures 22 and 23). We always need to take finer mesh past the shock formation (see Figures 24 and 25).

The last computation concerns a nonsymmetric solution. The aim of our simulations is to show that we recover good qualitative properties, even for nonradially symmetric data. We consider here as initial datum a Maxwell distribution with two temperatures, \( \theta_1 = 0.05 \) and \( \theta_2 = 0.015 \), for the initial amplitude given by

\[
a_0 = \frac{1}{\sqrt{2 \pi \sqrt{\theta_1 \theta_2}}} \exp \left( -\frac{(x - 0.5)^2}{2\theta_1} - \frac{(y - 0.5)^2}{2\theta_2} \right).
\]

The initial phase is reduced to \( \phi_0 = 0 \) (note that from (1.11), \( \partial_t \phi_{t=0} = -|a_0|^2 \neq 0 \), so \( \phi \) becomes instantaneously nontrivial). The scaled Planck factor \( \varepsilon \) is equal to \( \varepsilon = 0.005 \) and the simulations are performed with \( J = 2048 \) intervals in each direction. We plot in Figures 26, 27, and 28 the contour plots of \( \rho \varepsilon \), \( |j\varepsilon| \), and \( \text{Re}(u\varepsilon) \) at times \( t = 0.035 \) and \( t = 0.08 \). The oscillatory nature is enhanced at time \( t = 0.08 \) mainly in the \( y \) direction.

We recover the good behavior of physical observables before the formation time.
(a) Error w.r.t. $J = 2^M$

(b) Error w.r.t. $\varepsilon$

**Fig. 23.** $\text{err}_J (t = 0.05)$ for AP scheme.

(a) Error w.r.t. $J = 2^M$

(b) Error w.r.t. $\varepsilon$

**Fig. 24.** $\text{err}_\rho (t = 0.13)$ for AP scheme.

(a) Error w.r.t. $J = 2^M$

(b) Error w.r.t. $\varepsilon$

**Fig. 25.** $\text{err}_J (t = 0.13)$ for AP scheme.
Fig. 26. Contour plots of particle density $\rho^\varepsilon$ for $\varepsilon = 0.005$.

(a) $\rho^\varepsilon(t = 0.035)$

(b) $\rho^\varepsilon(t = 0.08)$

Fig. 27. Contour plots of current density $|s^\varepsilon|$ for $\varepsilon = 0.005$.

(a) $|s^\varepsilon|(t = 0.035)$

(b) $|s^\varepsilon|(t = 0.08)$

Fig. 28. Contour plots of $\text{Re}(u^\varepsilon)$ for $\varepsilon = 0.005$.

(a) $\text{Re}(u^\varepsilon)(t = 0.035)$

(b) $\text{Re}(u^\varepsilon)(t = 0.08)$
of singularities. To distinguish the evolution for each direction \((x, y)\), we present slices of physical observables along the lines \(x = 0.5\) and \(y = 0.5\). We compare the solutions computed by the Strang splitting scheme and our AP scheme at time \(t = 0.035\) in Figures 29 and 30. We cannot notice any differences. Past the formation time of singularities, some noticeable differences can be seen on \(x = 0.5\) and \(y = 0.5\) plane slices (see Figures 31 and 32).

4. Extension to other frameworks.

4.1. The linear case: Global solution of the viscous eikonal equation.

The advantage of introducing an artificial diffusion is more striking in the linear case. Consider

\[
    i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = V_{\text{ext}} u^\varepsilon; \quad u^\varepsilon|_{t=0} = a_0 e^{i\phi_0/\varepsilon},
\]

with \(V_{\text{ext}} = V_{\text{ext}}(t,x)\) real-valued. As noticed in [22], using the Madelung transform in this context is interesting only if vacuum can be avoided. Having the idea of Grenier
Fig. 31. Slices of $\rho^\varepsilon(t = 0.08)$ for Strang and AP schemes.

Fig. 32. Slices of $|J^\varepsilon|(t = 0.08)$ for Strang and AP schemes.

[29] in mind, the counterpart of (1.8) reads as

\[
\begin{align*}
\partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon &= i\varepsilon \Delta a^\varepsilon, \\
\partial_t \phi^\varepsilon + \frac{1}{2} |\nabla \phi^\varepsilon|^2 + V_{\text{ext}} &= 0,
\end{align*}
\]

(a) $x = 0.5$ \hspace{1cm} (b) $x = 0.5$

We note that the two equations are decoupled: The second equation is of Hamilton–Jacobi type (known as the eikonal equation in this context), and the solution must be expected to develop singularities in finite time (see, e.g., [12]), so this approach is doomed to fail for large time. On the other hand, introducing the artificial diffusion as in (1.11), we obtain

\[
\begin{align*}
\partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon &= i\varepsilon \Delta a^\varepsilon - i\varepsilon a^\varepsilon \Delta \phi^\varepsilon, \\
\partial_t \phi^\varepsilon + \frac{1}{2} |\nabla \phi^\varepsilon|^2 + V_{\text{ext}} &= \varepsilon^2 \Delta \phi^\varepsilon,
\end{align*}
\]

The system is still decoupled, but the good news is that the diffusion introduced into the Hamilton–Jacobi equation smooths out the solution. Therefore, this system can
be considered as a good candidate to obtain an AP scheme for the linear Schrödinger equation, regardless of the presence of vacuum.

Instead of giving full details of the analogues of Theorems 4, 7, and 8, as well as of their proofs, we shall simply give a functional framework where the viscous eikonal equation can be solved globally in time. Since we consider the case \( \varepsilon > 0 \) only, we drop out the \( \varepsilon^2 \) term in the viscous eikonal equation and consider

\[
\partial_t \phi_{\text{eik}} + \frac{1}{2} |\nabla \phi_{\text{eik}}|^2 + V_{\text{ext}} = \Delta \phi_{\text{eik}}; \quad \phi_{\text{eik}}(0, x) = \phi_0(x).
\]

For \( k \geq 1 \), let

\[
\mathcal{SL}_k = \{ f \in C^k(\mathbb{R}^d; \mathbb{R}), \quad \partial^\alpha f \in L^\infty(\mathbb{R}^d) \forall 1 \leq |\alpha| \leq k \}.
\]

For \( k \geq 2 \), let

\[
\mathcal{SQ}_k = \{ f \in C^k(\mathbb{R}^d; \mathbb{R}), \quad \partial^\alpha f \in L^\infty(\mathbb{R}^d) \forall 2 \leq |\alpha| \leq k \}.
\]

The key result is the following lemma.

**Lemma 13.** Let \( k \geq 2 \) and \( \phi_0 \in \mathcal{SL}_k \), \( V_{\text{ext}} \in L^\infty_{\text{loc}}(\mathbb{R}^d; \mathcal{SL}_k) \). Then (4.3) has a unique solution \( \phi_{\text{eik}} \in C(\mathbb{R}^+; \mathcal{SL}_{k-1}) \).

**Remark 14.** No such global result must be expected in the larger class \( \mathcal{SQ}_k \).

Indeed, suppose \( d = 1 \), \( V_{\text{ext}} = 0 \), and \( \phi_0(x) = -x^2/2 \). If (4.3) had a solution \( \phi_{\text{eik}} \in C(\mathbb{R}^+; \mathcal{SL}_2) \), then \( w = \partial^2_x \phi_{\text{eik}} \in C(\mathbb{R}^+; L^\infty) \) would solve

\[
\partial_t w + v \partial_x w + w^2 = \partial^2_x w; \quad w(0, x) = -1.
\]

The solution of this equation does not depend on \( x \). It is given by \( w(t, x) = 1/(t - 1) \), hence a contradiction. However, in the periodic case \( x \in \mathbb{T} \), this technical issue disappears (see Remark 15).

**Proof.** The first step consists of constructing the gradient of \( \phi_{\text{eik}} \). Differentiating (4.3) in space, we have to solve

\[
\partial_t v + v \cdot \nabla v + \nabla V_{\text{ext}} = \Delta v; \quad v(0, x) = \nabla \phi_0(x).
\]

This equation is solved locally in time by a fixed point argument by using Duhamel’s formula

\[
v(t) = e^{t \Delta} \nabla \phi_0 - \int_0^t e^{(t-s) \Delta} (v \cdot \nabla v)(s) ds - \int_0^t e^{(t-s) \Delta} (\nabla V_{\text{ext}})(s) ds.
\]

The right-hand side is a contraction on

\[
\{ w \in C \left( [0, T]; W^{k-1, \infty} \right), \quad \| w \|_{L^\infty([0, T]; W^{k-1, \infty})} \leq 2 \| \nabla \phi_0 \|_{W^{k-1, \infty}} \},
\]

provided that \( T > 0 \) is sufficiently small. This follows from the properties of the heat kernel: for \( t > 0 \),

\[
\| e^{t \Delta} f \|_{L^\infty} \leq \| f \|_{L^\infty}; \quad \| e^{t \Delta} \nabla f \|_{L^\infty} \leq \frac{C}{\sqrt{t}} \| f \|_{L^\infty}.
\]

The solution to (4.4) is then global, thanks to the maximum principle (see, e.g., [48, Proposition 52.8]), which implies that \( v_- \leq v \leq v_+ \), where

\[
\partial_t v_{\pm} + \| \nabla V_{\text{ext}} \|_{L^\infty} = 0; \quad v_{\pm}(0, x) = \pm \| \nabla \phi_0 \|_{L^\infty}.
\]
Once such a solution \( v \in C(\mathbb{R}_+; W^{k-1, \infty}) \) is constructed, set

\[
\phi_{\text{eik}}(t) = \phi_0 - \int_0^t \left( \frac{1}{2} \left| v(s) \right|^2 + V_{\text{ext}}(s) - \text{div} v(s) \right) ds.
\]

We check that \( \partial_t (\nabla \phi_{\text{eik}} - v) = \nabla \partial_t \phi_{\text{eik}} - \partial_t v = 0 \). Therefore, \( v = \nabla \phi_{\text{eik}} \), and \( \phi_{\text{eik}} \) solves (4.3). Rewriting (4.3) as

\[
\partial_t \phi_{\text{eik}} + \frac{1}{2} \left| v \right|^2 + V_{\text{ext}} = \Delta \phi_{\text{eik}}; \quad \phi_{\text{eik}}(0, x) = \phi_0(x),
\]

Duhamel’s formula reads as

\[
\phi_{\text{eik}}(t) = e^{t \Delta} \phi_0 - \int_0^t e^{(t-s) \Delta} \left( \left| v(s) \right|^2 \right) ds - \int_0^t e^{(t-s) \Delta} V_{\text{ext}}(s) ds,
\]

from which we conclude that \( \phi_{\text{eik}} \in C(\mathbb{R}_+; \mathcal{S}L_{k-1}) \).

Remark 15. In the periodic setting \( x \in \mathbb{T}^d = (\mathbb{R}/(2\pi \mathbb{Z}))^d \), it is natural to work in the Wiener algebra

\[
W = \left\{ f : \mathbb{T}^d \to \mathbb{C}, \quad f(x) = \sum_{j \in \mathbb{Z}^d} b_j e^{ij \cdot x} \text{ with } (b_j)_{j \in \mathbb{Z}^d} \in l^1(\mathbb{Z}^d) \right\},
\]

and for \( k \geq 0 \),

\[
W^k = \left\{ f : \mathbb{T}^d \to \mathbb{C}, \quad \partial^\alpha f \in W \ \forall |\alpha| \leq k \right\}.
\]

Then Lemma 13 is easily adapted by replacing \( \mathcal{S}L_k \) with \( W^k \).

4.2. Other nonlinearities. One might believe that Theorem 8 is bound to the cubic one-dimensional Schrödinger equation, which is completely integrable, and that the two aspects are related. We show that this is not the case, since Theorem 8 remains valid for other nonlinearities (with \( d = 1 \)). Consider now

\[
(i\varepsilon \partial_t + \frac{\varepsilon^2}{2} \Delta) u^\varepsilon = f \left( |u^\varepsilon|^2 \right) u^\varepsilon, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d.
\]

We suppose that there exists \( \delta > 0 \) such that \( f'(y) \geq \delta \) for all \( y \geq 0 \).

Following [29], the only modification we have to make to recover the results in section 2.1 consists of replacing the symmetrizer (2.2) with

\[
S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{4f'(a_1^2 + a_2^2)} I_d \end{pmatrix}.
\]

From our assumption on \( f \), \( S \) and its inverse \( S^{-1} \) are uniformly bounded, provided that \( a \) is bounded. As a matter of fact, the exact assumption in [29] is \( f' > 0 \), and all the results in section 2.1 remain valid under this assumption.

The more precise assumption \( f''(y) \geq \delta \) becomes useful to prove that in dimension \( d = 1 \), the solution is global. The main point to notice is that the conservation of mass and momentum are the same as for (1.1), like in Proposition 1. On the other hand, the conservation of energy becomes

\[
\frac{d}{dt} \left( \| \varepsilon \nabla u^\varepsilon(t) \|^2_{L^2} + \int_{\mathbb{R}^d} F \left( |u^\varepsilon(t, x)|^2 \right) dx \right) = 0,
\]
where $F$ is an antiderivative of $f$,

$$F(y) = \int_0^y f(r)dr.$$ 

By assumption,

$$F(y) \geq \frac{\delta}{2} y^2 + f(0)y,$$

so the potential energy controls the $L^4$-norm:

$$\|u^\varepsilon(t)\|_{L^4}^4 \leq \frac{2}{\delta} \left( \int_{\mathbb{R}^d} F \left( |u^\varepsilon(t,x)|^2 \right) dx - f(0) \|u^\varepsilon(t)\|_{L^2}^2 \right),$$

where we have used the conservation of mass in the last inequality. This implies an estimate of the form

$$\|\varepsilon \nabla u^\varepsilon(t)\|_{L^2}^2 + \|u^\varepsilon(t)\|_{L^4}^4 \leq C,$$

with $C$ independent of $t \geq 0$ and $\varepsilon \in (0,1]$. Therefore, the analysis presented in section 2.3 can be repeated line by line.

**Example 16.** The assumption $f' \geq \delta > 0$ is satisfied in the following cases:

- Cubic-quintic nonlinearity: $f(y) = y + \lambda y^2$, $\lambda \geq 0$.
- Cubic plus saturated nonlinearity: $f(y) = \delta y + \eta y^{1+\lambda}$, $\delta, \eta, \lambda > 0$.

**REFERENCES**


